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The Continuity between Logic and Metalogic from the Perspective of Logical Geometry

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- Aristotelian diagrams (e.g. square of oppositions):
  - long and rich history in philosophical logic
  - in recent years:
    - used in epistemic logic, free logic, dynamic logic, connexive logic, etc.
    - used in linguistics, cognitive science, artificial intelligence, law, etc.

  $\Rightarrow$ Aristotelian diagrams as *lingua franca*

- logical geometry
  - not just: develop new applications of Aristotelian diagrams
  - but mainly: study them as objects of independent interest
  - Aristotelian diagrams give rise to various logico-linguistic phenomena:
    - lexicalization patterns
    - logic-sensitivity of Aristotelian diagrams
    - Boolean subtypes of Aristotelian diagrams
    - complementarities between Aristotelian diagrams
    - interaction with duality diagrams

- until recently:
  - mainly object-logical Aristotelian diagrams
  - scattered throughout the literature, some metalogical diagrams (Béziau, Löbner, Seuren)
- aim of this talk: present a general framework
  - develop new metalogical Aristotelian diagrams
  - unifying perspective on existing metalogical Aristotelian diagrams
  - all the logico-linguistic phenomena known from object-logical Aristotelian diagrams also arise for metalogical Aristotelian diagrams

    $\Rightarrow$ continuity between object- and metalogical Aristotelian diagrams
- this talk is based on joint work with Hans Smessaert

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- the Aristotelian relations (in a Boolean logical system S): φ and ψ are

S-contradictory	iff	S $\models \neg(\varphi \wedge \psi)$	and	S $\models \neg(\neg\varphi \wedge \neg\psi)$
S-contrary	iff	S $\models \neg(\varphi \wedge \psi)$	and	S $\not\models \neg(\neg\varphi \wedge \neg\psi)$
S-subcontrary	iff	S $\not\models \neg(\varphi \wedge \psi)$	and	S $\models \neg(\neg\varphi \wedge \neg\psi)$
in S-subalternation	iff	S $\models \varphi \rightarrow \psi$	and	S $\not\models \psi \rightarrow \varphi$

- this can be generalized to an arbitrary Boolean algebra $\mathbb{B}$: x and y are

$\mathbb{B}$-contradictory	iff	$x \wedge_{\mathbb{B}} y = \bot_{\mathbb{B}}$	and	$x \vee_{\mathbb{B}} y = \top_{\mathbb{B}}$
$\mathbb{B}$-contrary	iff	$x \wedge_{\mathbb{B}} y = \bot_{\mathbb{B}}$	and	$x \vee_{\mathbb{B}} y \neq \top_{\mathbb{B}}$
$\mathbb{B}$-subcontrary	iff	$x \wedge_{\mathbb{B}} y \neq \bot_{\mathbb{B}}$	and	$x \vee_{\mathbb{B}} y = \top_{\mathbb{B}}$
in $\mathbb{B}$-subalternation	iff	$x \wedge_{\mathbb{B}} y = x$	and	$x \wedge_{\mathbb{B}} y \neq y$

- this subsumes both object- and metalogical uses:
  - object-logical: let $\mathbb{B}$ be $\mathbb{B}(\mathsf{S})$ (Lindenbaum-Tarski algebra of S)
  - metalogical: let $\mathbb{B}$ be $\wp(\mathbb{B}(\mathsf{S}))$ or $\wp(\mathbb{B}(\mathsf{S}) \times \mathbb{B}(\mathsf{S}))$

- the opposition relations: φ and ψ are

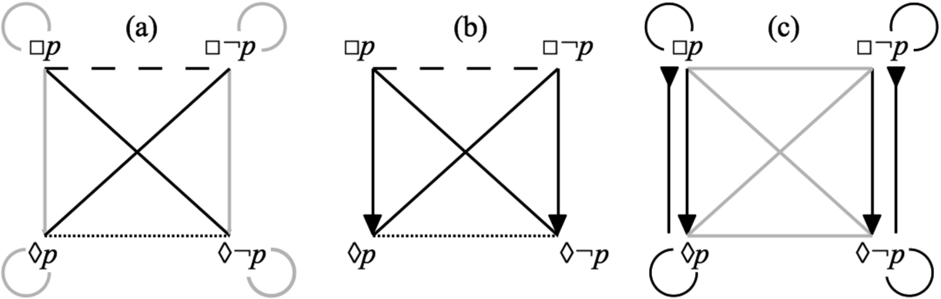
S-contradictory	iff	$\mathsf{S} \models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \models \neg(\varphi \wedge \psi)$
S-contrary	iff	$\mathsf{S} \models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \not\models \neg(\varphi \wedge \psi)$
S-subcontrary	iff	$\mathsf{S} \not\models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \models \neg(\varphi \wedge \psi)$
S-noncontradictory	iff	$\mathsf{S} \not\models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \not\models \neg(\varphi \wedge \psi)$

- the implication relations: φ and ψ are

in S-bi-implication	iff	$\mathsf{S} \models \varphi \rightarrow \psi$	and	$\mathsf{S} \models \psi \rightarrow \varphi$
in S-left-implication	iff	$\mathsf{S} \models \varphi \rightarrow \psi$	and	$\mathsf{S} \not\models \psi \rightarrow \varphi$
in S-right-implication	iff	$\mathsf{S} \not\models \varphi \rightarrow \psi$	and	$\mathsf{S} \models \psi \rightarrow \varphi$
in S-non-implication	iff	$\mathsf{S} \not\models \varphi \rightarrow \psi$	and	$\mathsf{S} \not\models \psi \rightarrow \varphi$

- motivation:
  - disentangling the Aristotelian relations into opposition and implication
  - the Aristotelian relations are informationally optimal between the opposition and implication relations (Smessaert & Demey 2014)

contradiction	*CD*	*bi-implication*	*BI*
contrariety	*C*	*left-implication*	*LI*
subcontrariety	*SC*	*right-implication*	*RI*
non-contradiction	*NCD*	*non-implication*	*NI*



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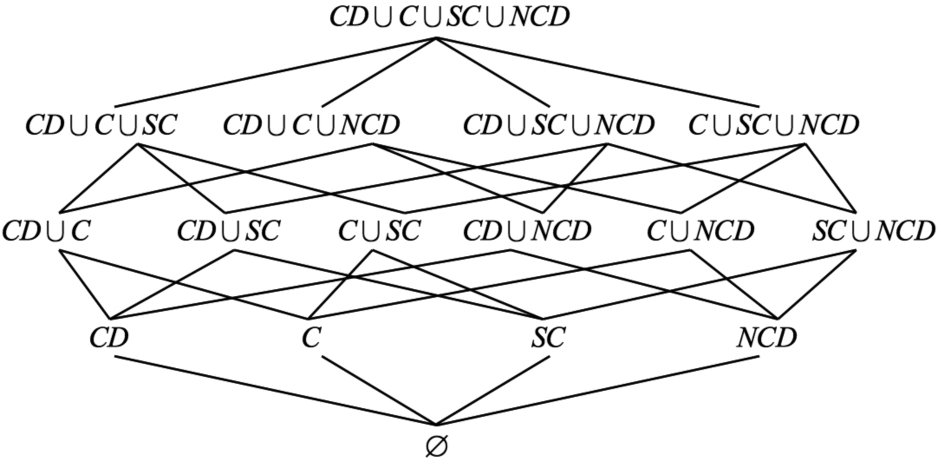
- logical system S (often left implicit)

- easy: every pair of formulas stands in exactly one opposition relation
- the opposition relations form a partition of $\mathbb{B}(\mathsf{S}) \times \mathbb{B}(\mathsf{S})$
- the opposition relations can be viewed as atoms in a Boolean algebra
  - the elements of this Boolean algebra are $\bigcup \mathcal{X}$, for $\mathcal{X} \subseteq \{CD, C, SC, NCD\}$
  - it has $2^4 = 16$ elements
  - its bottom and top elements are $\emptyset$ and $CD \cup C \cup SC \cup NCD = \mathbb{B}(\mathsf{S}) \times \mathbb{B}(\mathsf{S})$

- visualizations of this Boolean algebra:
  - Hasse diagram: 2D or 3D rhombic dodecahedron (RDH)
  - Aristotelian diagram: rhombic dodecahedron

  (close connection between Hasse RDH and Aristotelian RDH)


$CD \cup C \cup SC \cup NCD$
$CD \cup C \cup SC$
$CD \cup C \cup NCD$
$CD \cup SC \cup NCD$
$C \cup SC \cup NCD$
$CD \cup C$
$CD \cup SC$
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$C \cup NCD$
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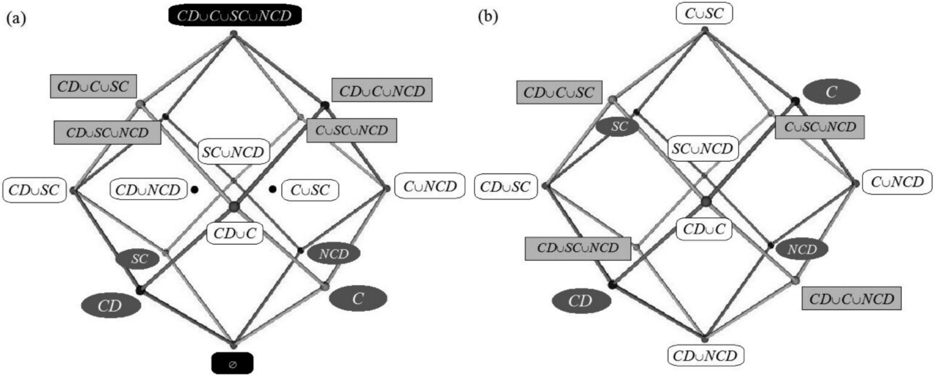
CD
C
SC
NCD
$\varnothing$


(a)
CD∪C∪SC∪NCD
CD∪C∪SC
CD∪C∪NCD
CD∪SC∪NCD
C∪SC∪NCD
SC∪NCD
CD∪SC
CD∪NCD
C∪SC
C∪NCD
CD∪C
SC
NCD
CD
C
⌀
(b)
C∪SC
CD∪C∪SC
C
SC
C∪SC∪NCD
SC∪NCD
CD∪SC
C∪NCD
CD∪C
CD∪SC∪NCD
NCD
CD
CD∪C∪NCD
CD∪NCD

- Aristotelian RDH for the opposition relations
  $\Rightarrow$ largest metalogical diagram so far!
- there are many object-logical Aristotelian RDHs
  - e.g. propositional logic, modal logic S5, public announcement logic, etc.
  - 4-partition of $\mathbb{B}(\mathsf{S}) \times \mathbb{B}(\mathsf{S})$ vs. 4-partition of $\mathbb{B}(\mathsf{S})$

OG	CD	C	SC	NCD
CPL	$p \wedge q$	$p \wedge \neg q$	$\neg p \wedge q$	$\neg p \wedge \neg q$
S5	$\Box p$	$p \wedge \Diamond \neg p$	$\neg p \wedge \Diamond p$	$\Box \neg p$
PAL	$\langle !p \rangle K q$	$\neg p \wedge K[!p]q$	$\neg p \wedge \neg K[!p]q$	$\langle !p \rangle \neg K q$

- the internal structure of (object-logical) RDH has been extensively studied (Béziau, Moretti, Smessaert, LD):
  - it contains 18 classical Aristotelian squares
  - it contains 3 degenerate Aristotelian squares
  - it contains 4 weak Jacoby-Sesmat-Blanché hexagons
  - it contains 6 strong JSB hexagons
  - it contains 12 Sherwood-Czezowski hexagons
  - it contains 6 Buridan octagons
  - complementarity strong JSB hexagons/Buridan octagons

$\Rightarrow$ all these properties straightforwardly carry over from the object-logical to the metalogical RDH

- strong and weak notions of (sub)contrariety: φ and ψ are

strongly S-*contrary*	iff	$\mathsf{S} \models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \not\models \varphi \vee \psi$
weakly S-*contrary*	iff	$\mathsf{S} \models \neg(\varphi \wedge \psi)$		
strongly S-*subcontrary*	iff	$\mathsf{S} \not\models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \models \varphi \vee \psi$
weakly S-*subcontrary*	iff			$\mathsf{S} \models \varphi \vee \psi$

- Humberstone: "traditionalist approach" vs "modernist approach"

- connection with the opposition relations:

$$C_s = C \qquad SC_s = SC$$
$$C_w = CD \cup C \qquad SC_w = CD \cup SC$$

- note that $CD = C_w \cap SC_w$

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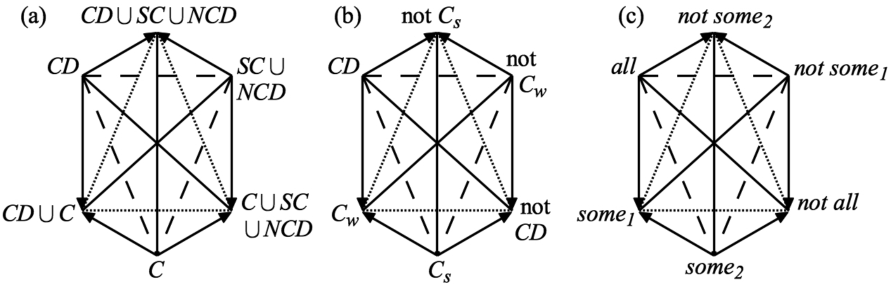

- pragmatic perspective:
  - $\langle CD, C_w \rangle$ forms a Horn scale
  - saying C_w triggers the scalar implicature not-CD
  - total meaning becomes: C_w but not CD, i.e. C_s
- lexicalization perspective:
  - co-lexicalization of weak and strong contrariety
  - cf. co-lexicalization of unilateral and bilateral *some* (Seuren & Jaspers)

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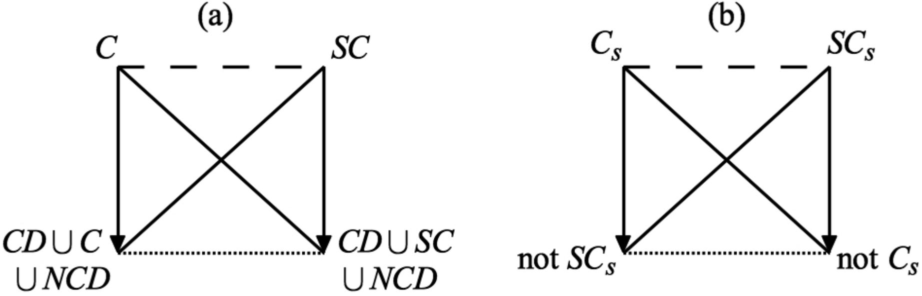
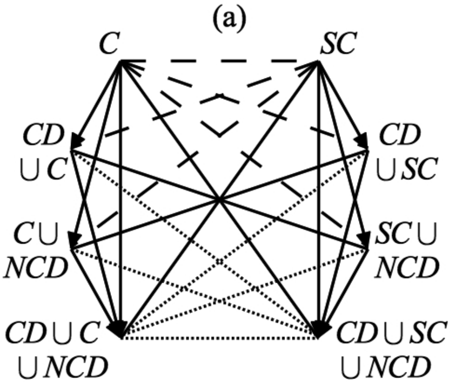
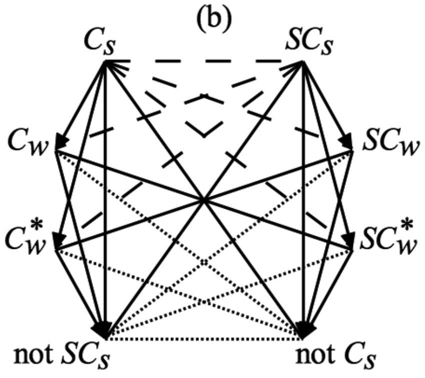
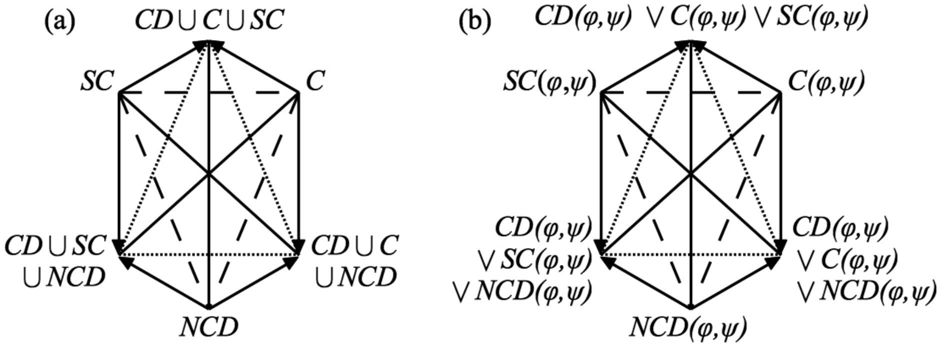


- the subalternation from C_s to not-SC_s can be split up by putting C_w in between
- the subalternation from SC_s to not-C_s can be split up by putting SC_w in between

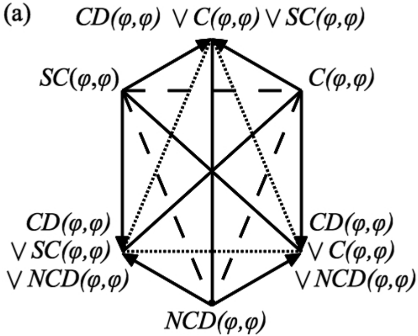

(a)
C
SC
CD ∪ C
CD ∪ SC
C ∪ NCD
SC ∪ NCD
CD ∪ C ∪ NCD
CD ∪ SC ∪ NCD

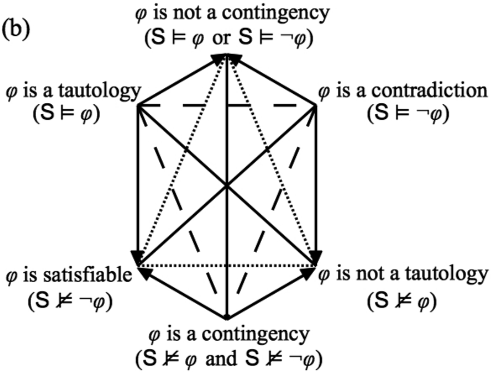

(b)
C_s
SC_s
C_w
SC_w
C_w^*
SC_w^*
not SC_s
not C_s

- in terms of relations
- in terms of statements about formulas φ, ψ



- what happens if we fill in the same formula twice (i.e. $\varphi = \psi$)?
  (in terms of relations, this corresponds to replacing R with $R \cap \Delta_{\mathbb{B}(\mathsf{S})}$)
- we obtain a hexagon for some well-known metalogical notions (Béziau)





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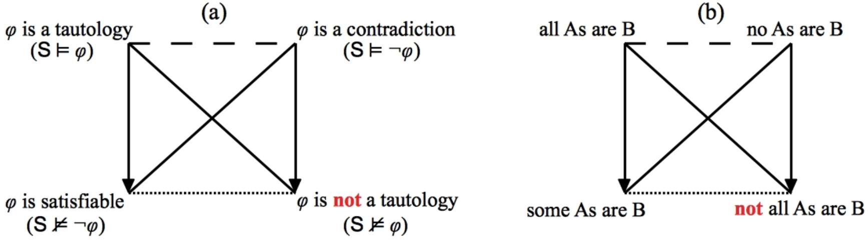
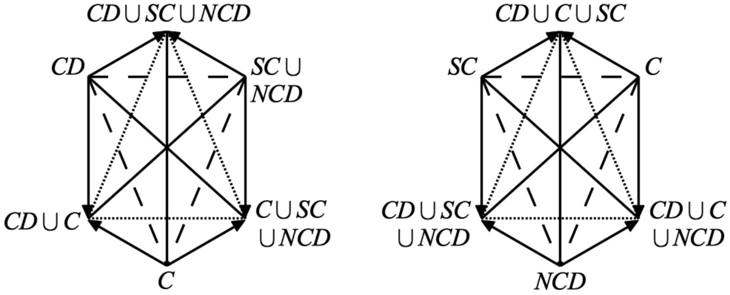
- lexicalization: tautology, satisfiable, contradictory vs non-tautology
- non-lexicalization of the O-corner of the square (Horn)
- analogues at the object-language level:
    - all, some, no vs not-all
    - necessary, possible, impossible vs not-necessary
    - always, sometimes, never vs not-always

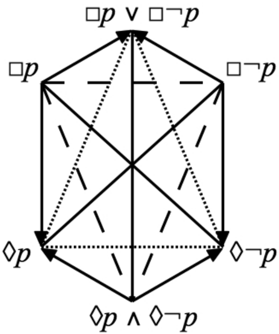


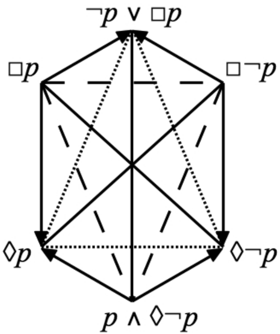


- isomorphic qua Aristotelian diagrams (both are JSB hexagons) (same configuration of Aristotelian relations)
- yet: Boolean differences
  - the left one is closed under the Boolean operations — strong JSB
  - the right one is not — weak JSB

⇒ a given type of Aristotelian diagram (e.g. JSB hexagon) can have various Boolean subtypes (e.g. strong/weak)

- this phenomenon is well-known from object-logical Aristotelian diagrams
- example: strong vs weak JSB hexagon in the modal logic D





- the implication relations closely resemble the opposition relations

$$\begin{array}{rcl} CD(\varphi,\psi) & \text{iff} & BI(\varphi,\neg\psi) \\ C(\varphi,\psi) & \text{iff} & LI(\varphi,\neg\psi) \\ SC(\varphi,\psi) & \text{iff} & RI(\varphi,\neg\psi) \\ NCD(\varphi,\psi) & \text{iff} & NI(\varphi,\neg\psi) \end{array}$$

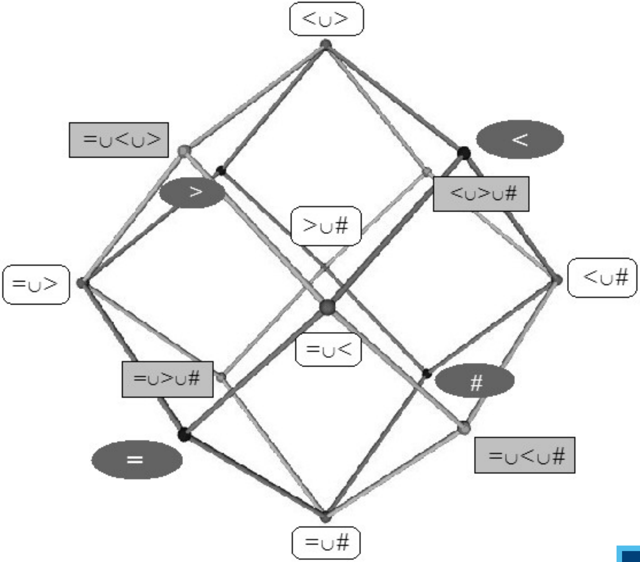
- the implication relations form a partition of $\mathbb{B}(\mathsf{S}) \times \mathbb{B}(\mathsf{S})$

$\Rightarrow$ atoms of a Boolean algebra
$\Rightarrow$ Hasse RDH for this Boolean algebra
$\Rightarrow$ Aristotelian RDH for this Boolean algebra
$\Rightarrow$ study the subdiagrams of this Aristotelian RDH

$\vdots$

- consider an arbitrary partial order $\leq$ on some set X
- some notions:
  - $x < y :\Leftrightarrow (x \leq y \text{ and } x \neq y)$
  - $x > y :\Leftrightarrow (x \geq y \text{ and } x \neq y)$
  - $x\#y :\Leftrightarrow \text{not}(x < y \text{ or } x > y)$
- easy to show: $=, <, >, \#$ form a partition of $S \times S$
- if $\leq$ happens to be the $\models$-relation on $\mathbb{B}(\mathsf{S})$:

$$
\begin{array}{lll}
= & \text{corresponds to} & BI \\
< & \text{corresponds to} & LI \\
> & \text{corresponds to} & RI \\
\# & \text{corresponds to} & NI
\end{array}
$$



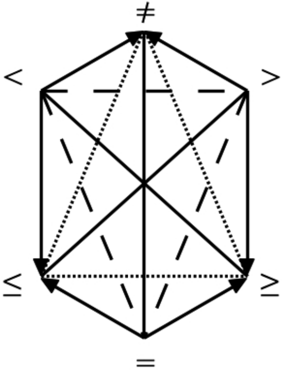
- from partial order to total order:
    - impose the additional axiom of totality: $\forall x, y \in S : x \leq y$ or $x \geq y$
    - equivalently, impose the assumption that $\# = \emptyset$

- effect on the Aristotelian RDH: pairwise collapses:

RDH		collapse	collapse		RDH
$=$	$\rightarrow$	$=$	$< \cup >$	$\leftarrow$	$< \cup > \cup \#$
$= \cup \#$	$\nearrow$			$\nwarrow$	$< \cup >$
$<$	$\rightarrow$	$<$	$= \cup >$	$\leftarrow$	$= \cup > \cup \#$
$< \cup \#$	$\nearrow$			$\nwarrow$	$= \cup >$
$>$	$\rightarrow$	$>$	$= \cup <$	$\leftarrow$	$= \cup < \cup \#$
$> \cup \#$	$\nearrow$			$\nwarrow$	$= \cup <$
$\#$	$\rightarrow$	$[\emptyset]$	$[= \cup < \cup >]$	$\leftarrow$	$= \cup < \cup >$
$[\emptyset]$	$\nearrow$			$\nwarrow$	$[= \cup < \cup > \cup \#]$

From Partial to Total Orders

- the Aristotelian RDH collapses into a strong JSB hexagon
- this hexagon was already known by Blanché (= the 'B' in 'JSB')



Logic-sensitivity

- this is an example of the ‘logic-sensitivity’ of Aristotelian diagrams
- well-known from object-logical Aristotelian diagrams (e.g. modal logic)

- sensitivity to the underlying axiomatization of the ordering relation:
  - partial order: 4-partition: $=$ / $>$ / $<$ / $\#$ RDH
  - total order: 3-partition: $=$ / $>$ / $<$ JSB

- sensitivity to the underlying modal logic:
  - modal logic K: 4-partition: $\Diamond\top \wedge \Box p$ / $\Diamond p \wedge \Diamond\neg p$ / $\Diamond\top \wedge \Box\neg p$ / $\Box\bot$ RDH
  - modal logic D: 3-partition: $\Box p$ / $\Diamond p \wedge \Diamond\neg p$ / $\Box\neg p$ JSB

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- Aristotelian = hybrid between opposition/implication
  $\Rightarrow$ some Aristotelian diagrams for opposition/implication relations can also be viewed as Aristotelian diagrams for the Aristotelian relations

  (e.g. Buridan octagon for strong/weak (sub)contrariety)

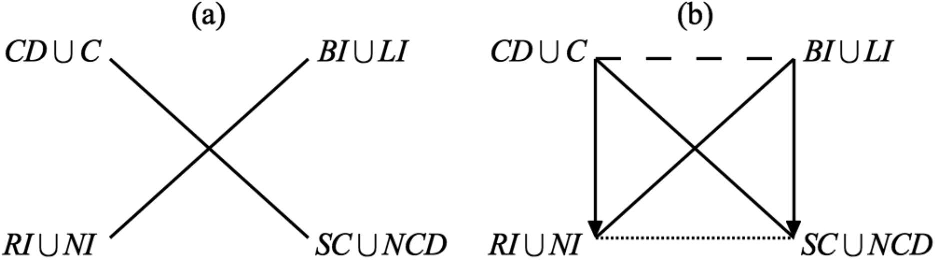
- but: in each of these diagrams:
  - *either* only opposition relations
  - *or* only implication relations

- now: diagrams that contain both opposition and implication relations

- already in the 80s, Löbner claimed that the following four relations form an Aristotelian square:

compatibility	$\not\models \neg(\varphi \wedge \psi)$	$SC \cup NCD$
implication	$\models \varphi \rightarrow \psi$	$BI \cup LI$
contrariety	$\models \neg(\varphi \wedge \psi)$	$CD \cup C$
non-implication	$\not\models \varphi \rightarrow \psi$	$RI \cup NI$

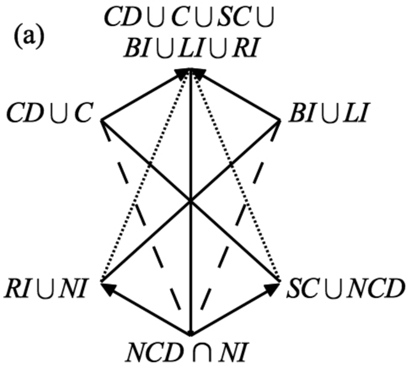
- note that these are *weak* opposition and implication relations: $SC_w^*, LI_w, C_w, RI_w^*$
- these four indeed form a square, but this square is
  - *classical* iff the relations' first argument (φ) is assumed to be satisfiable
  - *degenerated* otherwise (Béziau: "an X of opposition")



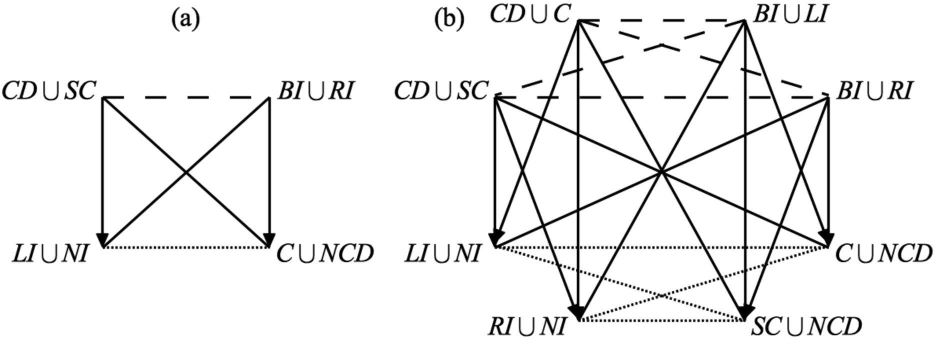
- another illustration of the logic-sensitivity of Aristotelian diagrams
- object-logical example: the four categorical statements form a
  - *classical* square iff the subject term is assumed to be non-empty
  - *degenerate* square otherwise

- first argument is satisfiable // first term has existential import

- Seuren (2014): 6 relations, forming a JSB hexagon
  ⇒ translate into opposition/implication terminology
  - a *JSB* hexagon iff the relations' first argument is satisfiable
  - a *U4* (= partially degenerated JSB) hexagon otherwise



- recall Löbner's relations:
  - 4 weak opposition/implication relations: $SC_w^*, LI_w, C_w, RI_w^*$
  - classical square iff $\varphi \neq \bot$
- completely analogously:
  - 4 other weak opposition/implication relations: $SC_w, LI_w^*, C_w^*, RI_w$
  - classical square iff $\varphi \neq \top$
- combination of these two squares:
  - all 8 weak opposition/implication relations together
  - minimal assumption: contingency of φ ($\varphi \neq \bot$ and $\varphi \neq \top$)
  - interesting if we also assume contingency of ψ
- this is the metalogical analogue of a well-known object-logical octagon
  - syllogistics with subject negation
  - Keynes, Johnson, Hacker, Reichenbach


(a)
CD∪SC
BI∪RI
LI∪NI
C∪NCD
(b)
CD∪C
BI∪LI
CD∪SC
BI∪RI
LI∪NI
C∪NCD
RI∪NI
SC∪NCD

- construct metalogical Aristotelian diagrams
  (in a mathematically precise sense)

- observed various connections and phenomena:
  - 4-partition of opposition relations, giving rise to Aristotelian RDH
  - internal structure of RDH (e.g. subdiagrams, complementarities)
  - unifying perspective on earlier work (e.g. Béziau, Seuren, Löbner)
  - lexicalization patterns (e.g. strong/weak contrariety)
  - logic-dependence (e.g. satisfiability of first argument)
  - Boolean subtypes of Aristotelian diagrams (e.g. strong/weak JSB)

- these are the counterparts of similar (and well-studied) connections and phenomena for object-logical Aristotelian diagrams

$\Rightarrow$ fundamental continuity between
object- and metalogical Aristotelian diagrams!

Thank you!

Lorenz Demey & Hans Smessaert,
Metalogical Decorations of Logical Diagrams,
Logica Universalis, forthcoming.

www.logicalgeometry.org