The Logical Geometry of Russell's Theory of Definite Descriptions

Lorenz Demey*

Abstract

This paper studies Russell's theory of definite descriptions (TDD) from the perspective of logical geometry, i.e. by focusing on the various Aristotelian diagrams it gives rise to. Russell analyzed sentences of the form 'the A is B' in terms of existence, uniqueness and universality conditions. I first show that each definite description gives rise to four distinct formulas (depending on negation scope), which jointly constitute a classical square of opposition. Next, I discuss the interplay between the Aristotelian square for TDD and that for the categorical statements. After arguing that the latter is already implicitly present in the former, I integrate both into a single Aristotelian diagram, viz. a so-called 'Buridan octagon'. Finally, I study the exact role of the existence and uniqueness conditions within TDD, by introducing two new logical systems based on these conditions, and showing that this has drastic consequences for the aforementioned Buridan octagon.

1 Introduction

Bertrand Russell was notoriously critical of Aristotle and his followers, writing, for example, that "Aristotle, in spite of his reputation, is full of absurdities" (Russell, 1950, p. 99), and that "[t]hroughout modern times, practically every advance in science, in logic, or in philosophy has had to be made in the teeth of opposition from Aristotle's disciples" (Russell, 1946, p. 237). Some of his most severe criticisms were directed at Aristotle's logical theories, which he considered "wholly false, with the exception of the formal theory of the syllogism, which is unimportant" (Russell, 1946, p. 237). One of Russell's objections against Aristotelian logic concerns its assumption of *existential import*, which would allow one to erroneously infer the (false) conclusion 'some mountains are golden' from the (true) premises 'all golden mountains are mountains' and 'all golden mountains are golden' (Russell, 1946, p. 232).

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Throughout the history of Aristotelian logic, the square of opposition has been used to visualize the relations of contradiction, (sub)contrariety and subalternation holding between the four categorical statements. In the past few years, there has been a revived interest in the square and its various extensions and reinterpretations, mostly under the label of *logical geometry* (Smessaert and Demey, 2014a, 2015). Although Russell does not seem to have discussed the square in any detail (Jager, 1972, p. 144), his criticisms of syllogistics are highly relevant here, too: if the assumption of existential import is no longer made, then the categorical square loses its (sub)contrarieties and subalternations.

It seems plausible that the particular severity of Russell's opposition to the tradition of Aristotelian logic is at least partially due to the fact that some of his own most important philosophical achievements went against exactly this tradition. Consider, for example, his statement that "[e]ven at the present day, all Catholic teachers of philosophy and many others still obstinately reject the discoveries of modern logic, and adhere with a strange tenacity to a system which is as definitely antiquated as Ptolemaic astronomy" (Russell, 1946, p. 230).[1] The "discoveries of modern logic" that Russell had in mind here probably include his logicist project (Whitehead and Russell, 1910, 1912, 1913) and his theory of definite descriptions (Russell, 1905). Especially the latter has been extremely influential in logic, philosophy of language and linguistics (Neale, 2005), and Russell and his contemporaries were already well aware of its significance. For example, Frank P. Ramsey praised Russell's theory of definite descriptions as a "paradigm of philosophy" (1931, p. 263), while Russell himself claimed *On Denoting* to be "his finest philosophical essay" (1956, p. 39).

In this paper I will argue that despite these apparent tensions, there exists a fruitful interaction between Aristotelian diagrams (as studied in logical geometry) and Russell's theory of definite descriptions (henceforth abbreviated as 'TDD'). Throughout the paper it will be emphasized that this interaction has advantages for both parties involved.

On the one hand, the systematic construction of Aristotelian diagrams for TDD will lead us to consider various formulas involving definite descriptions (and logical equivalences between such formulas) that have not been investigated in any detail in the literature on TDD so far. Furthermore, these Aristotelian diagrams yield a natural connection with some linguistic-cognitive considerations that are highly relevant for TDD, e.g. concerning the nature of natural language negation. Finally, the use of some well-chosen Aristotelian diagrams can be of great help in explaining the features and consequences of TDD, which can be particularly relevant in educational contexts involving students who do not have a strong background in formal logic, but do have extensive knowledge of Aristotelian syllogistics.[2]

On the other hand, TDD is a rich source of philosophically interesting decorations for a wide variety of Aristotelian diagrams (e.g. the classical square, but also hexagons, octagons, etc.). Furthermore, these applications illustrate several key techniques and phenomena that are studied in logical geometry (e.g. Boolean closures, bitstring semantics,

[1] In a similar vein: "Ever since the beginning of the seventeenth century, almost every serious intellectual advance has had to begin with an attack on some Aristotelian doctrine; in logic, this is still true at the present day" (Russell, 1946, p. 237). For the role of Catholicism in the debate beween modern logic and Aristotelian syllogistics, see Jaspers and Seuren (2016).

[2] Although this specific type of educational context is probably less widespread now than it was a few decades ago, it still occurs quite frequently; see Demey (2017) for a partial explanation.

the interplay between Aristotelian and Boolean structure, logic-sensitivity, etc.). Finally, certain Aristotelian diagrams for TDD turn out to be intimately related to—and thus shed important new light on—other Aristotelian diagrams that are well-known in logical geometry, but *prima facie* have nothing to do with definite descriptions.

The paper is organized as follows. Section 2 provides a brief overview of Russell's TDD and its later developments, focusing on those aspects that will be most relevant for our present purposes. Section 3 then introduces some of the key notions and techniques that are used in logical geometry to study Aristotelian diagrams. The next two sections develop and discuss various Aristotelian diagrams for TDD, and thus constitute the core of the paper. Section 4 first explains how TDD naturally gives rise to an Aristotelian square of opposition, and then shows that this square can be extended to an Aristotelian hexagon. Section 5 discusses the connection between the Aristotelian square for TDD and the traditional Aristotelian square for syllogistics. After arguing that the latter is already implicitly present in the former, both squares are integrated into a single Aristotelian diagram, viz. an octagon, which is then shown to be closely related to other well-known Aristotelian diagrams. Section 6, finally, wraps things up and summarizes the main results obtained in this paper.

2 Russell's Theory of Definite Descriptions

In this section I will give a brief overview of Russell's original TDD and some of its later developments. The focus will be on those aspects of TDD that are most relevant for the present paper, which means that some key topics—such as Donnellan's (1966) distinction between referential and attributive uses of definite descriptions—will not be discussed in any detail.[3]

Definite descriptions are expressions of the form 'the so-and-so', such as 'the President of the United States' and 'the man standing over there'. They can occur in the subject position of a sentence (e.g. 'the President of the United States is visiting France tomorrow') as well as in the predicate position (e.g. 'Barack Obama is the President of the United States'), but we will focus on the former. According to Russell's TDD, a sentence of the form 'the A is B' can be translated into the language of first-order logic (FOL) as follows:

$$\exists x\big(Ax \wedge \forall y(Ay \rightarrow y = x) \wedge Bx\big).$$

For ease of notation, we will henceforth often make use of Neale's (1990) *restricted quantifier* notation, and abbreviate 'the A is B' as [the $x\colon Ax]Bx$, which should be read as: 'the unique x such that x is A, is B'. However, it should be emphasized that nothing of philosophical substance hinges on this notational convention. It can be shown that the first-order formalization of [the $x\colon Ax]Bx$ stated above is equivalent to the conjunction of the following three formulas:

$$\begin{array}{ll} (\textsc{ex}) & \exists x Ax, \\ (\textsc{un}) & \forall x \forall y\big((Ax \wedge Ay) \rightarrow x = y\big), \\ (\textsc{uv}) & \forall x(Ax \rightarrow Bx). \end{array}$$

[3]For more comprehensive overviews of the literature, see Neale (1990), Reimer and Bezuidenhout (2004) and Ludlow (2013); furthermore, see Elbourne (2013) for a recent alternative account.

These three formulas jointly express the truth conditions of [the $x\colon Ax]Bx$. First of all, the *existence* condition $(\textsc{ex})$ states that there exists *at least one* A; secondly, the *uniqueness* condition $(\textsc{un})$ states that there exists *at most one* A; and finally, the *universality* condition $(\textsc{uv})$ states that *all* As are B. Note that $(\textsc{ex})$ and $(\textsc{un})$ together express that there exists *exactly one* A. Furthermore, since [the $x\colon Ax]Bx$ is equivalent to the conjunction $(\textsc{ex}) \wedge (\textsc{un}) \wedge (\textsc{uv})$, it trivially entails each of these conditions. Contrapositively, if at least one of these three conditions fails, then [the $x\colon Ax]Bx$ itself is also false.

Much of the subsequent literature on TDD has focused on one of these three conditions. For example, the exact linguistic status of $(\textsc{ex})$ has been widely debated. As was explained above, Russell's TDD takes $(\textsc{ex})$ to be a part of the *truth conditions* of [the $x\colon Ax]Bx$, and hence, if $(\textsc{ex})$ is false, then [the $x\colon Ax]Bx$ will also be false. By contrast, Strawson (1950, 1964) has argued that $(\textsc{ex})$ is rather a *presupposition* of [the $x\colon Ax]Bx$, and hence, if $(\textsc{ex})$ is false, then [the $x\colon Ax]Bx$ does not have a truth value at all.

Next, there has been an extensive discussion of the so-called problem of *incomplete descriptions*, i.e. definite descriptions for which $(\textsc{un})$ fails. For example, according to a literal interpretation of Russell's TDD, a sentence such as 'the book is on the shelf' entails that there exists at most one book in the entire universe, which seems ludicrous. Various refinements of TDD have been proposed in order to solve this problem; for example, *ellipsis* theories hold that an incomplete description [the $x\colon Ax]$ can always be enriched into a description [the $x\colon Ax \wedge Rx]$ for which $(\textsc{un})$ does hold (Vendler, 1967), while theories of *quantifier domain restriction* hold that the quantifiers in $(\textsc{un})$ do not range over the entire universe, but rather over a contextually determined, restricted domain of discourse (Stanley and Szabó, 2000). Another, more radical solution consists in dropping $(\textsc{un})$ from the truth conditions of definite descriptions altogether, and recovering it by means of pragmatic principles (Szabó, 2000; Demey, 2009).

Finally, $(\textsc{uv})$ plays a prominent role in theories that go beyond standard *singular* definite descriptions, and also try to account for definite descriptions involving *plurals* (e.g. 'the wives of King Henry VIII') or *mass nouns* (e.g. 'the water in the Dead Sea'). It has been argued that these descriptions, too, satisfy a version of $(\textsc{uv})$. For example, the sentence 'the wives of King Henry VIII were pale' means that *all* wives of King Henry VIII were pale, and similarly, the sentence 'the water in the Dead Sea is salty' means that *all* water in the Dead Sea is salty (Sharvy, 1980; Brogaard, 2007).

3 Logical Geometry

In this section I will give a broad outline of logical geometry. The focus will again be on those notions and techniques that will be of direct use later in the paper, and consequently, certain other topics—such as the visual-geometric perspective on Aristotelian diagrams (Demey and Smessaert, 2014, 2016a,c; Smessaert and Demey, 2016)—will not be discussed in any detail.

Given a logical system S that has the usual Boolean connectives and a model-theoretical semantics, the *Aristotelian relations for* S are defined as follows (Smessaert and Demey, 2014a): two formulas $\varphi, \psi \in \mathcal{L}_{\mathsf{S}}$ are

contradictory in S	iff	$\mathsf{S} \models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \models \varphi \vee \psi$,
contrary in S	iff	$\mathsf{S} \models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \not\models \varphi \vee \psi$,
subcontrary in S	iff	$\mathsf{S} \not\models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \models \varphi \vee \psi$,
in subalternation in S	iff	$\mathsf{S} \models \varphi \rightarrow \psi$	and	$\mathsf{S} \not\models \psi \rightarrow \varphi$.

These clauses are straightforward formalizations of the more traditional definitions of the Aristotelian relations. For example, for φ and ψ to be contrary in S, it is required that there exists no S-model $\mathbb{M}$ such that $\mathbb{M} \models \varphi \wedge \psi$ (i.e. φ and ψ cannot be true together), and that there does exist an S-model $\mathbb{M}$ such that $\mathbb{M} \not\models \varphi \vee \psi$ (i.e. φ and ψ can be false together). Note that in the contemporary definition, the Aristotelian relations are defined *relative to a logical system* S, and hence it is possible for formulas to stand in different Aristotelian relations in different logical systems (Demey, 2015). It is also easy to show that within a single logical system, the Aristotelian relations are defined *up to logical equivalence*, i.e. if $\varphi \equiv_{\mathsf{S}} \varphi'$ and $\psi \equiv_{\mathsf{S}} \psi'$, then the pairs (φ, ψ) and (φ', ψ') stand in exactly the same Aristotelian relation in S.

Given a logical system S as above and a finite set $\mathcal{F}$ of contingent and pairwise non-equivalent formulas,[4] an *Aristotelian diagram for* $\mathcal{F}$ *in* S is a diagram that visualizes the formulas of $\mathcal{F}$ and all the Aristotelian relations holding between these formulas, using the visual code shown in Figure 1(a). Typical examples of Aristotelian diagrams for classical propositional logic (CPL) include the *classical square of opposition*, the *degenerate square*, the *Jacoby-Sesmat-Blanché (JSB) hexagon* and the *Buridan octagon* shown in Figures 1(b), 1(c), 1(d) and 2(a), respectively.[5] These diagrams all belong to different Aristotelian families, since there exist no Aristotelian relation-preserving bijections between them (Demey and Smessaert, 2017a).[6] The requirement that the formulas appearing in Aristotelian diagrams be contingent and pairwise non-equivalent is made for systematic as well as historical reasons, which are discussed in more detail in Smessaert and Demey (2014a). Because the notions of contingency and equivalence are defined relative to the logical system S (cf. Footnote 4), this requirement forms another source of logic-dependence in Aristotelian diagrams (next to the logic-dependence of the Aristotelian relations themselves). Finally, it should be noted that many authors also assume that Aristotelian diagrams are closed under negation, i.e. if $\varphi \in \mathcal{F}$ then $\neg\varphi \equiv_{\mathsf{S}} \psi$ for some $\psi \in \mathcal{F}$ (Demey and Smessaert, 2016c). All Aristotelian diagrams discussed in this paper indeed satisfy this additional requirement.

It is often interesting to study the subdiagrams of a given Aristotelian diagram (Smessaert and Demey, 2014b). Focusing on Aristotelian diagrams that are closed under negation (and thus have an even number of formulas), an easy combinatorial argument shows that an Aristotelian diagram consisting of $2n$ formulas contains exactly $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ Aristotelian subdiagrams of $2m$ formulas (for $m \leq n$). For example, the JSB hexagon in Figure 1(d) has $6 = 2 \times 3$ formulas, and thus contains $\binom{3}{2} = 3$ subdiagrams of $2 \times 2 = 4$ formulas, i.e. 3 square subdiagrams, which are shown in Figures 1(b), 2(b) and 2(c).

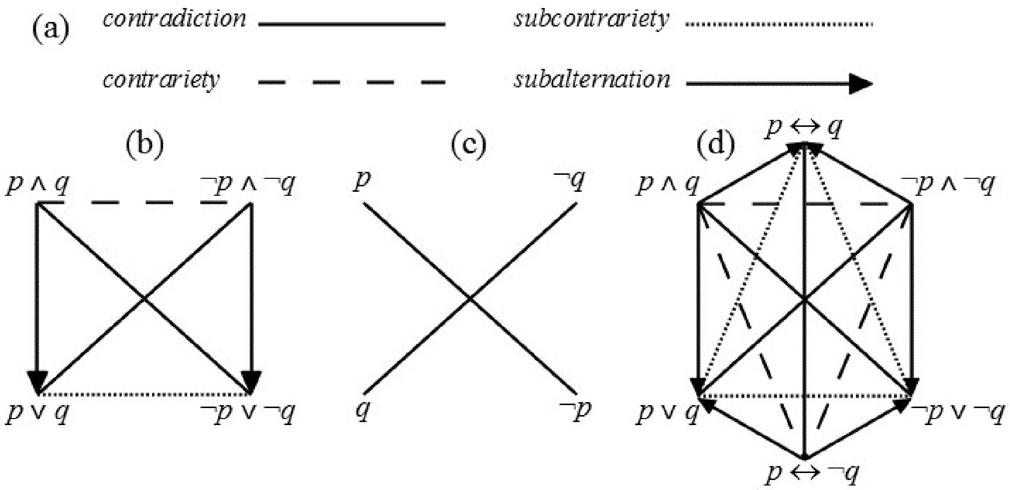
An Aristotelian diagram is said to be *Boolean closed* iff it contains every contingent

[4]So $\mathsf{S} \not\models \varphi$, $\mathsf{S} \not\models \neg\varphi$ and $\varphi \not\equiv_{\mathsf{S}} \psi$ for all distinct formulas $\varphi, \psi \in \mathcal{F}$.

[5]Many families of Aristotelian diagrams are named after the authors who first made use of them; see Smessaert and Demey (2014a) for some historical background and bibliographic references.

[6]For example, letting $\mathcal{F}$ and $\mathcal{F}'$ be the sets of formulas appearing in resp. Figures 1(b) and 1(c), there exists no bijection $f\colon \mathcal{F} \rightarrow \mathcal{F}'$ such that $R(\varphi, \psi)$ iff $R(f(\varphi), f(\psi))$ for all formulas $\varphi, \psi \in \mathcal{F}$ and Aristotelian relations R.

Figure 1: (a) Code for visualizing the Aristotelian relations, (b) classical square of opposition, (c) degenerate square and (d) JSB hexagon in CPL.



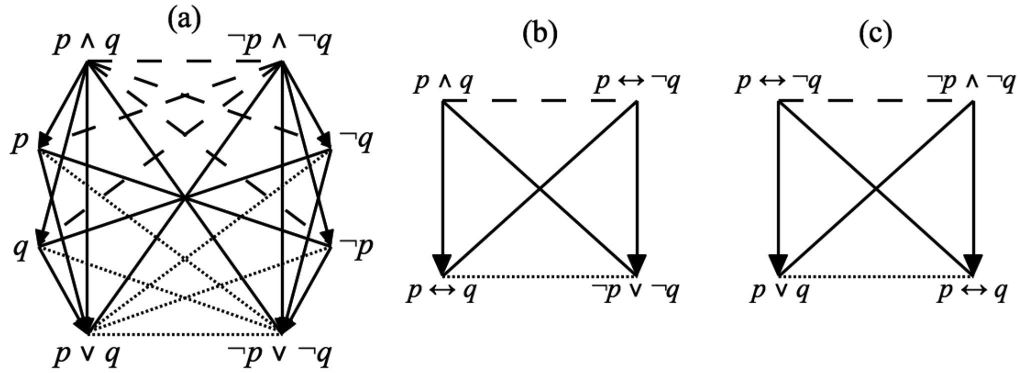
Boolean combination of its formulas (up to logical equivalence). The *Boolean closure* of an Aristotelian diagram is the smallest Boolean closed Aristotelian diagram containing it as a subdiagram. For example, the classical square in Figure 1(b) is not Boolean closed, since it contains the formulas $p \wedge q$ and $\neg p \wedge \neg q$, but not their Boolean combination $(p \wedge q) \vee (\neg p \wedge \neg q)$ (or any formula that is CPL-equivalent to it). One can easily show that the Boolean closure of this square is the JSB hexagon in Figure 1(d)—note that $(p \wedge q) \vee (\neg p \wedge \neg q) \equiv_{\mathsf{CPL}} p \leftrightarrow q$.

Logical geometry also makes extensive use of *bitstrings*, which are compact semantic representations of formulas that allow us to easily determine the Aristotelian relations holding between these formulas (Smessaert and Demey, 2017). A systematic technique for assigning a bitstring semantics to any finite set $\mathcal{F}$ of formulas in any logical system S is described in detail in Demey and Smessaert (2017a). We say that $\mathcal{F}$ *induces* (in S) the partition $\Pi_{\mathsf{S}}(\mathcal{F}) := \{\bigwedge_{\varphi \in \mathcal{F}} \pm\varphi\} - \{\bot\}$ (where $+\varphi = \varphi$ and $-\varphi = \neg\varphi$).[7] One can then show that every formula $\varphi \in \mathcal{F}$ is S-equivalent to a disjunction of $\Pi_{\mathsf{S}}(\mathcal{F})$-formulas, viz. $\varphi \equiv_{\mathsf{S}} \bigvee\{\alpha \in \Pi_{\mathsf{S}}(\mathcal{F}) \mid \mathsf{S} \models \alpha \to \varphi\}$. The bitstring representation of φ keeps track which formulas of $\Pi_{\mathsf{S}}(\mathcal{F})$ enter into this disjunction (and is thus essentially a disjunctive normal form). For example, if $\Pi_{\mathsf{S}}(\mathcal{F}) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, then φ is represented by the bitstring 1011 iff $\varphi \equiv_{\mathsf{S}} \alpha_1 \vee \alpha_3 \vee \alpha_4$. The number $|\Pi_{\mathsf{S}}(\mathcal{F})|$ is the number of bit positions, i.e. the *bitstring length*, that is minimally required to represent the formulas of $\mathcal{F}$ by means of bitstrings. One can show that the Boolean closure of $\mathcal{F}$ contains exactly $2^{|\Pi_{\mathsf{S}}(\mathcal{F})|}$ formulas, and hence $2^{|\Pi_{\mathsf{S}}(\mathcal{F})|} - 2$ *contingent* formulas. For example, letting $\mathcal{F}$ be the set of formulas appearing in the square in Figure 1(b), we have that $\Pi_{\mathsf{CPL}}(\mathcal{F}) = \{\alpha_1, \alpha_2, \alpha_3\} = \{p \wedge q, p \leftrightarrow \neg q, \neg p \wedge \neg q\}$. Since $p \vee q \equiv_{\mathsf{CPL}} \alpha_1 \vee \alpha_2$, the bitstring representation of $p \vee q$ is 110. Furthermore, the Boolean closure of $\mathcal{F}$ contains exactly $2^3 - 2 = 6$ contingent formulas, which are exactly those constituting the JSB hexagon in Figure 1(d).

The formal properties of bitstring semantics are very well understood; I will finish

[7] The set $\Pi_{\mathsf{S}}(\mathcal{F})$ is called a 'partition' because its elements are (i) jointly exhaustive ($\mathsf{S} \models \bigvee \Pi_{\mathsf{S}}(\mathcal{F})$), and (ii) mutually exclusive ($\mathsf{S} \models \neg(\alpha \wedge \beta)$ for distinct $\alpha, \beta \in \Pi_{\mathsf{S}}(\mathcal{F})$).

Figure 2: (a) Buridan octagon in CPL, (b–c) the two remaining classical squares embedded inside the JSB hexagon shown in Figure 1(d).



this section by mentioning some results from Demey and Smessaert (2017a) that will be useful later in this paper. First of all, one can show that if $\mathcal{F} \subseteq \mathcal{F}'$, then $\Pi_{\mathsf{S}}(\mathcal{F}')$ is a *refinement* of $\Pi_{\mathsf{S}}(\mathcal{F})$, i.e. for every $\alpha' \in \Pi_{\mathsf{S}}(\mathcal{F}')$, there exists an $\alpha \in \Pi_{\mathsf{S}}(\mathcal{F})$ such that $\mathsf{S} \models \alpha' \to \alpha$. Secondly, it holds that if $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$, then $\Pi_{\mathsf{S}}(\mathcal{F}) = \Pi_{\mathsf{S}}(\mathcal{F}_1) \wedge_{\mathsf{S}} \Pi_{\mathsf{S}}(\mathcal{F}_2) = \{\alpha \wedge \beta \mid \alpha \in \Pi_{\mathsf{S}}(\mathcal{F}_1), \beta \in \Pi_{\mathsf{S}}(\mathcal{F}_2), \alpha \wedge \beta \text{ is } \mathsf{S}\text{-consistent}\}$. Furthermore, using this characterization of $\Pi_{\mathsf{S}}(\mathcal{F})$, one can easily show that $\Pi_{\mathsf{S}}(\mathcal{F})$ is a refinement of $\Pi_{\mathsf{S}}(\mathcal{F}_1)$ as well as $\Pi_{\mathsf{S}}(\mathcal{F}_2)$. Thirdly and finally, if S' is a stronger logic than S, then for any set of formulas $\mathcal{F}$, it holds that $\Pi_{\mathsf{S}'}(\mathcal{F}) = \{\alpha \in \Pi_{\mathsf{S}}(\mathcal{F}) \mid \alpha \text{ is } \mathsf{S}'\text{-consistent}\}$. Note that the first two results are concerned with different sets of formulas ($\mathcal{F}, \mathcal{F}', \mathcal{F}_1, \mathcal{F}_2$) in a single logical system (S); by contrast, the third result is concerned with a single set of formulas ($\mathcal{F}$) in two different logical systems (S, S').

4 Basic Aristotelian Diagrams for Definite Descriptions

In this section we will start putting the topics presented in the previous two sections together, by constructing various Aristotelian diagrams for Russell's TDD. Subsection 4.1 explains how TDD naturally gives rise to an Aristotelian square of opposition. Next, in Subsection 4.2 the Boolean closure of this square is shown to be a JSB hexagon, and a bitstring semantics for both diagrams is introduced, with special attention to its linguistic-cognitive importance.

4.1 A Square of Opposition for Definite Descriptions

Russell (1905, pp. 484–405) famously argued that his theory of definite descriptions is able to solve a number of logico-linguistic 'puzzles', one of which concerns the law of excluded middle. When applied to a sentence of the form 'the A is B', this law seems to yield that either 'the A is B' is true or its Boolean negation, 'the A is not B', is true. However, when there exist no As,[8] then both statements seem to be false, thus yielding a violation of the law of excluded middle.

[8] Russell's own illustrious example involves A = 'the present King of France' (and B = 'bald').

In order to solve this puzzle, Russell argued that the natural language sentence 'the A is not B' is ambiguous, depending on what the scope of its negation is taken to be. Its two interpretations look as follows (in FOL and restricted quantifier format, respectively):

1. $\neg\exists x\big(Ax \wedge \forall y(Ay \rightarrow y = x) \wedge Bx\big)$, $\qquad \neg[\text{the } x\colon Ax]Bx$,
2. $\exists x\big(Ax \wedge \forall y(Ay \rightarrow y = x) \wedge \neg Bx\big)$, $\qquad [\text{the } x\colon Ax]\neg Bx$.

Note that the first interpretation, $\neg[\text{the } x\colon Ax]Bx$, is the 'real' Boolean negation of the original $[\text{the } x\colon Ax]Bx$, and these two can indeed be shown to satisfy the law of excluded middle, since at least one of them is always true. In particular, if the extension of A is empty, then $[\text{the } x\colon Ax]Bx$ is false, while $\neg[\text{the } x\colon Ax]Bx$ is true. As to the second interpretation, by contrast, one can show that if the extension of A is empty, then both $[\text{the } x\colon Ax]Bx$ and $[\text{the } x\colon Ax]\neg Bx$ are false; however, this does not constitute a violation of the law of excluded middle, since the latter is not the Boolean negation of the former. Summing up: neither of the two interpretations leads to a genuine violation of the law of excluded middle.

Let's now re-examine Russell's argument from the perspective of logical geometry. We have just seen above that $[\text{the } x\colon Ax]Bx$ and $\neg[\text{the } x\colon Ax]Bx$ satisfy the law of excluded middle, which can be formalized as follows: $\mathsf{FOL} \models [\text{the } x\colon Ax]Bx \vee \neg[\text{the } x\colon Ax]Bx$. These two formulas can also be shown to satisfy the principle of non-contradiction, i.e. $\mathsf{FOL} \models \neg([\text{the } x\colon Ax]Bx \wedge \neg[\text{the } x\colon Ax]Bx)$. Taken together, this means exactly that $[\text{the } x\colon Ax]Bx$ and $\neg[\text{the } x\colon Ax]Bx$ are *contradictory* to each other. Moving on to the second interpretation of 'the A is not B', we have also seen above that $[\text{the } x\colon Ax]Bx$ and $[\text{the } x\colon Ax]\neg Bx$ do not satisfy the law of excluded middle, i.e. $\mathsf{FOL} \not\models [\text{the } x\colon Ax]Bx \vee [\text{the } x\colon Ax]\neg Bx$. However, these two formulas do satisfy the principle of non-contradiction, i.e. $\mathsf{FOL} \models \neg([\text{the } x\colon Ax]Bx \wedge [\text{the } x\colon Ax]\neg Bx)$.[9] Taken together, this means exactly that $[\text{the } x\colon Ax]Bx$ and $[\text{the } x\colon Ax]\neg Bx$ are *contrary* to each other. In sum, then, the two interpretations of 'the A is not B' that were distinguished by Russell turn out to stand in different Aristotelian relations to the original sentence, 'the A is B'.[10]

Since the two interpretations of 'the A is not B' result from adding a negation in two different positions, an obvious extension concerns the formula that results from adding a negation in *both* of those positions. Together with the original (negationless) sentence, we thus obtain four formulas involving definite descriptions:

$$
\begin{array}{rl}
\exists x\big(Ax \wedge \forall y(Ay \rightarrow y = x) \wedge Bx\big), & [\text{the } x\colon Ax]Bx,\\
\neg\exists x\big(Ax \wedge \forall y(Ay \rightarrow y = x) \wedge Bx\big), & \neg[\text{the } x\colon Ax]Bx,\\
\exists x\big(Ax \wedge \forall y(Ay \rightarrow y = x) \wedge \neg Bx\big), & [\text{the } x\colon Ax]\neg Bx,\\
\neg\exists x\big(Ax \wedge \forall y(Ay \rightarrow y = x) \wedge \neg Bx\big), & \neg[\text{the } x\colon Ax]\neg Bx.
\end{array}
$$

[9] To see this, note that for any FOL-model $M = \langle D, I\rangle$ such that $M \models [\text{the } x\colon Ax]Bx \wedge [\text{the } x\colon Ax]\neg Bx$, there would exist an element $d \in D$ such that $I(A) = \{d\}$ and simultaneously $d \in I(B)$ and $d \notin I(B)$, which is impossible.

[10] This observation is certainly not new; for example, it was already made by Haack (1965, p. 65) and Brown (1984, p. 315–316), and more recently also by Speranza and Horn (2010, p. 281; 2012, p. 134) and Martin (2016, p. 676). Sayward (1993, p. 38–39) also makes use of the Aristotelian notions of contradiction and contrariety in his discussion of Russell's TDD, but without making the specific point that we are currently considering.

As to the Aristotelian relations holding between these four formulas, we have already seen above that [the $x\colon Ax]Bx$ is contradictory to $\neg$[the $x\colon Ax]Bx$. Completely analogously, one can also show that [the $x\colon Ax]\neg Bx$ is contradictory to $\neg$[the $x\colon Ax]\neg Bx$. Furthermore, we have also seen above that [the $x\colon Ax]Bx$ is contrary to [the $x\colon Ax]\neg Bx$, which means that (i) and (ii) hold:

(i) $\mathsf{FOL} \models \neg([\text{the } x\colon Ax]Bx \wedge [\text{the } x\colon Ax]\neg Bx)$,
(ii) $\mathsf{FOL} \not\models [\text{the } x\colon Ax]Bx \vee [\text{the } x\colon Ax]\neg Bx$.

Using elementary Boolean principles such as De Morgan's laws, these conditions can equivalently be expressed as follows:

(i′) $\mathsf{FOL} \models \neg[\text{the } x\colon Ax]Bx \vee \neg[\text{the } x\colon Ax]\neg Bx)$,
(i″) $\mathsf{FOL} \models [\text{the } x\colon Ax]Bx \to \neg[\text{the } x\colon Ax]\neg Bx$,
(i‴) $\mathsf{FOL} \models [\text{the } x\colon Ax]\neg Bx \to \neg[\text{the } x\colon Ax]Bx)$,

(ii′) $\mathsf{FOL} \not\models \neg(\neg[\text{the } x\colon Ax]Bx \wedge \neg[\text{the } x\colon Ax]\neg Bx)$,
(ii″) $\mathsf{FOL} \not\models \neg[\text{the } x\colon Ax]\neg Bx \to [\text{the } x\colon Ax]Bx$,
(ii‴) $\mathsf{FOL} \not\models \neg[\text{the } x\colon Ax]Bx \to [\text{the } x\colon Ax]\neg Bx$.

Conditions (i′) and (ii′) jointly express that $\neg$[the $x\colon Ax]Bx$ is subcontrary to $\neg$[the $x\colon Ax]\neg Bx$. Similarly, (i″) and (ii″) jointly express that there is a subalternation from [the $x\colon Ax]Bx$ to $\neg$[the $x\colon Ax]\neg Bx$, while (i‴) and (ii‴) jointly express that there is a subalternation from [the $x\colon Ax]\neg Bx$ to $\neg$[the $x\colon Ax]Bx$.[11]

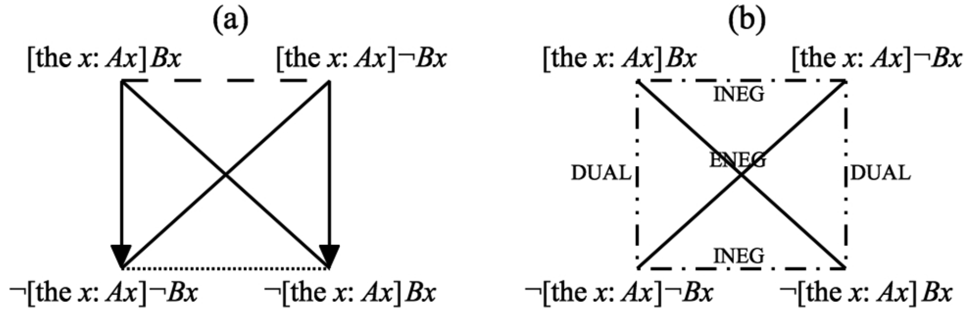
All of this can be summarized by stating that the four definite description formulas constitute a classical square of opposition, which is shown in Figure 3(a). It should be emphasized that although this square makes use of Aristotelian notions such as (sub)contrariety, it is entirely defined in ordinary first-order logic (FOL), and thus does not depend in any way on the assumption of existential import that Russell was so critical of. Furthermore, this square essentially captures all the central claims of Russell's account of negated definite description formulas. A sentence of the form 'the A is not B' is ambiguous between two distinct interpretations, both of which are incompatible with (i.e. cannot be true together with) 'the A is B'. For one interpretation, however, this incompatibility amounts to a full-blown contradiction, whereas for the other it amounts to a mere contrariety.

The Aristotelian square in Figure 3(a) also contains the formula $\neg$[the $x\colon Ax]\neg Bx$, which—to the best of my knowledge—has not been discussed in any detail in the literature on Russell's TDD so far.[12] Rather than being a mere artefact needed to 'complete' the square, however, this fourth formula is quite interesting from the perspective of TDD

[11] Given the subalternation from [the $x\colon Ax]\neg Bx$ to $\neg$[the $x\colon Ax]Bx$ (and the satisfiability of the former), there exist situations (formally: FOL-models) in which these two formulas are both true. One might use the existence of such situations to argue for the existence of a *third* interpretation of the sentence 'the A is not B' (distinct from both [the $x\colon Ax]\neg Bx$ and $\neg$[the $x\colon Ax]Bx$), but such an argument would be an example of what Neale (2007) has called a *scene-reading error*.

[12] A (logically equivalent formulation of a) special case of this formula can be found in the literature on principles of *self-predication* (Lambert, 2002; Heylen, 2016). In natural language, sentences of the form 'the A is A' sound plainly self-evident (e.g.: 'the teacher of Alexander the Great is a teacher of Alexander the Great'). However, the straightforward formalization [the $x\colon Ax]Ax$ is not a FOL-tautology, since it is false in cases where there does not exist exactly one A. By contrast, the weakened version $[(\text{EX}) \wedge (\text{UN})] \to$ [the $x\colon Ax]Ax$ is indeed a FOL-tautology. The latter formula turns out to be equivalent to $\neg$[the $x\colon Ax]\neg Ax$ (cf. infra).

Figure 3: (a) Aristotelian square for definite descriptions, (b) the corresponding duality square.



itself, since it turns out to express a *weak* version of [the $x\colon Ax]Bx$. To see this, recall that on Russell's tripartite analysis of definite description formulas, [the $x\colon Ax]Bx$ is equivalent to the conjunction $(\textsc{ex})\wedge(\textsc{un})\wedge(\textsc{uv})$ (see Section 2). Analogously, [the $x\colon Ax]\neg Bx$ is equivalent to the conjunction $(\textsc{ex})\wedge(\textsc{un})\wedge(\textsc{uv*})$, with $(\textsc{uv}^*)$ expressing the *negative universality* condition $\forall x(Ax \to \neg Bx)$. Hence, $\neg$[the $x\colon Ax]\neg Bx$ itself corresponds to $\neg[(\textsc{ex})\wedge(\textsc{un})\wedge(\textsc{uv*})]$, which is equivalent to $[(\textsc{ex})\wedge(\textsc{un})] \to \neg(\textsc{uv*})$, and hence also to $[(\textsc{ex})\wedge(\textsc{un})] \to [(\textsc{un})\wedge\neg(\textsc{uv*})]$. Now, since $\neg(\textsc{uv*})$ is equivalent to $\exists x(Ax \wedge Bx)$, it follows that $(\textsc{un}) \wedge \neg(\textsc{uv*})$ is equivalent to $\exists x(Ax \wedge \forall y(Ay \to y = x) \wedge Bx)$, i.e. to [the $x\colon Ax]Bx$. Putting everything together, this means that $\neg$[the $x\colon Ax]\neg Bx$ is equivalent to

$$[(\textsc{ex}) \wedge (\textsc{un})] \to [\text{the } x\colon Ax]Bx.$$

This shows that $\neg$[the $x\colon Ax]\neg Bx$ expresses that the A is B, *conditional upon* there being exactly one A.[13] Hence, if there exists exactly one A, then [the $x\colon Ax]Bx$ and $\neg$[the $x\colon Ax]\neg Bx$ always have the same truth value (regardless of the extension of B); by contrast, if there does not exist exactly one A, then [the $x\colon Ax]Bx$ is always false, whereas $\neg$[the $x\colon Ax]\neg Bx$ is always true.[14]

Finally, it should be noted that the four definite description formulas not only constitute an *Aristotelian square*, but also a *duality square*, which is shown in Figure 3(b).[15] For example, the internal negation (INEG) of [the $x\colon Ax]Bx$ is [the $x\colon Ax]\neg Bx$, its external negation (ENEG) is $\neg$[the $x\colon Ax]Bx$, and finally, its dual (DUAL) is $\neg$[the $x\colon Ax]\neg Bx$. Although Aristotelian squares and duality squares are sometimes confused with each other (D'Alfonso, 2012; Mélès, 2012), in recent years, it has been argued extensively that the Aristotelian relations and the duality relations are conceptually independent from each other (Löbner, 2011; Demey, 2012b; Smessaert, 2012; Westerståhl, 2012; Demey and Smessaert, 2017b, 2016b). Despite this conceptual independence, however, the fact remains that many of the most well-known Aristotelian squares also happen to be duality squares—for example, the squares for modal logic, propositional logic, syllogistics, etc.

[13]Note that if $\neg$[the $x\colon Ax]\neg Bx$ is reformulated as $[(\textsc{ex}) \wedge (\textsc{un})] \to$ [the $x\colon Ax]Bx$, then the subalternation from [the $x\colon Ax]Bx$ to $\neg$[the $x\colon Ax]\neg Bx$ in the square in Figure 3(a) becomes trivial, since it takes on the form of a subalternation from φ to $\psi \to \varphi$.

[14]Similar remarks can be made about the right-hand-side of the square in Figure 3(a); in particular, $\neg$[the $x\colon Ax]Bx$ can be shown to be equivalent to $[(\textsc{ex}) \wedge (\textsc{un})] \to$ [the $x\colon Ax]\neg Bx$.

[15]The duality square in Figure 3(b) is drawn using the code for visualizing the duality relations that is also used in Demey and Smessaert (2017b, 2016b).

Figure 4: (a) JSB hexagon for definite descriptions, (b) alternative formulation in terms of the (EX) and (UN) conditions.

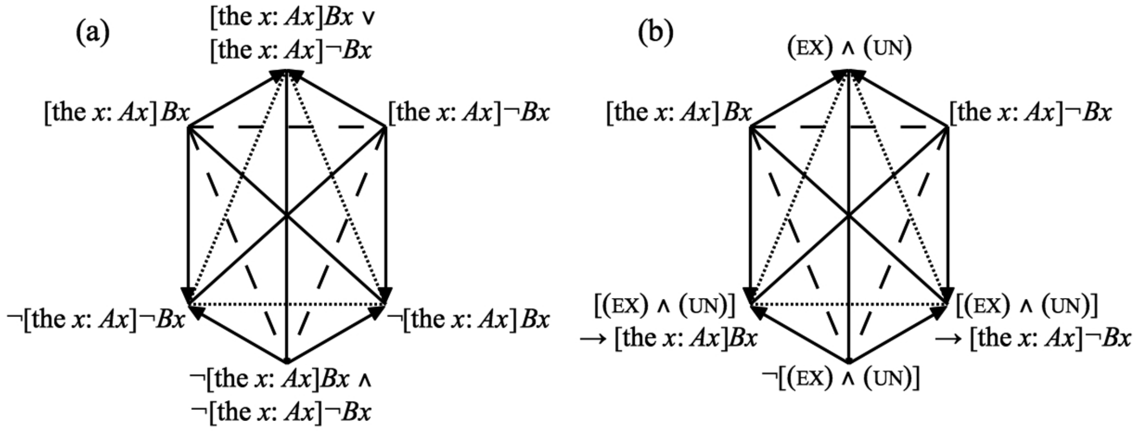


Figure 3 shows that the square for Russell's TDD can be added to this list.

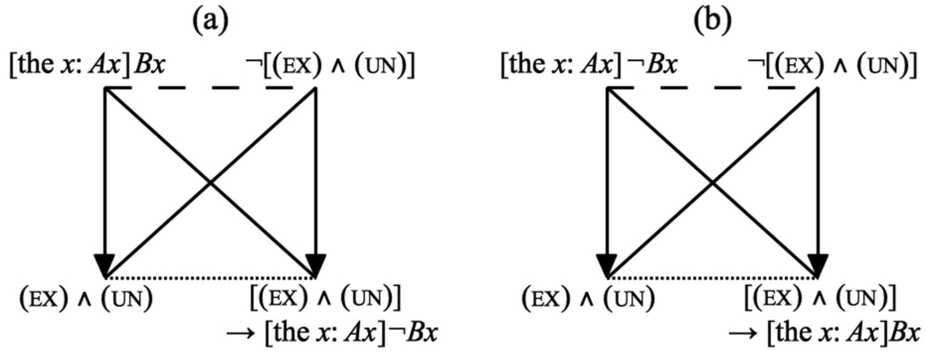
4.2 A JSB Hexagon for Definite Descriptions

As was already discussed in Section 3 in the context of CPL, Aristotelian squares of oppositions are not Boolean closed, and their Boolean closures are JSB hexagons. Returning to the context of Russell's TDD, we see that the definite description square in Figure 3(a) is not Boolean closed either, since it lacks the disjunction [the $x\colon Ax]Bx \vee$ [the $x\colon Ax]\neg Bx$ as well as the conjunction $\neg$[the $x\colon Ax]Bx \wedge \neg$[the $x\colon Ax]\neg Bx$. By adding these two formulas to the original square, we obtain its Boolean closure, viz. the JSB hexagon for definite descriptions shown in Figure 4(a).

One can easily show that the disjunction [the $x\colon Ax]Bx\vee$[the $x\colon Ax]\neg Bx$ is logically equivalent to (EX) $\wedge$ (UN); in other words, this disjunction expresses that there exists exactly one A. Recalling from Subsection 4.1 that $\neg$[the $x\colon Ax]Bx$ and $\neg$[the $x\colon Ax]\neg Bx$ can also be rewritten in terms of (EX) and (UN), this means that the JSB hexagon shown in Figure 4(a) can be reformulated as the one shown in Figure 4(b). This hexagon illustrates the importance of the (EX)- and (UN)-conditions in Russell's TDD; for example, the subalternations from the definite description formulas [the $x\colon Ax]Bx$ and [the $x\colon Ax]\neg Bx$ to (EX) $\wedge$ (UN) show that TDD treats (EX) and (UN) as part of those formulas' truth conditions—rather than as being their presuppositions, as in Schlenker's Transparancy theory (2007, p. 239).

We also saw in Section 3 that a JSB hexagon contains three square subdiagrams. The JSB hexagon for definite descriptions in Figure 4(b) first of all contains the square of which it is the Boolean closure, which was already shown in Figure 3(a) (modulo the (EX)/(UN) reformulation of $\neg$[the $x\colon Ax](\neg)Bx$). Its second and third square subdiagram are shown in Figures 5(a) and 5(b), respectively. These two squares show that the formulas [the $x\colon Ax]Bx$ and [the $x\colon Ax]\neg Bx$ are perfectly 'symmetric' with respect to the (EX)- and (UN)-conditions: by systematically substituting [the $x\colon Ax]\neg Bx$ for [the $x\colon Ax]Bx$ and vice versa, we can transform the square in Figure 5(a) into that of Figure 5(b) (and vice versa).

Figure 5: Two of the three Aristotelian squares that are embedded inside the JSB hexagon for definite descriptions.



We will now define the bitstring semantics for the Aristotelian square and hexagon for definite descriptions (since the latter is the Boolean closure of the former, they have the same bitstring semantics). By applying the technique described in Section 3, we obtain the partition Π^{FOL}_{TDD} consisting of the following formulas:

$$\begin{array}{lll} \alpha_1 & := & [\text{the } x\colon Ax]Bx, \\ \alpha_2 & := & [\text{the } x\colon Ax]\neg Bx, \\ \alpha_3 & := & \neg[(\text{EX}) \wedge (\text{UN})]. \end{array}$$

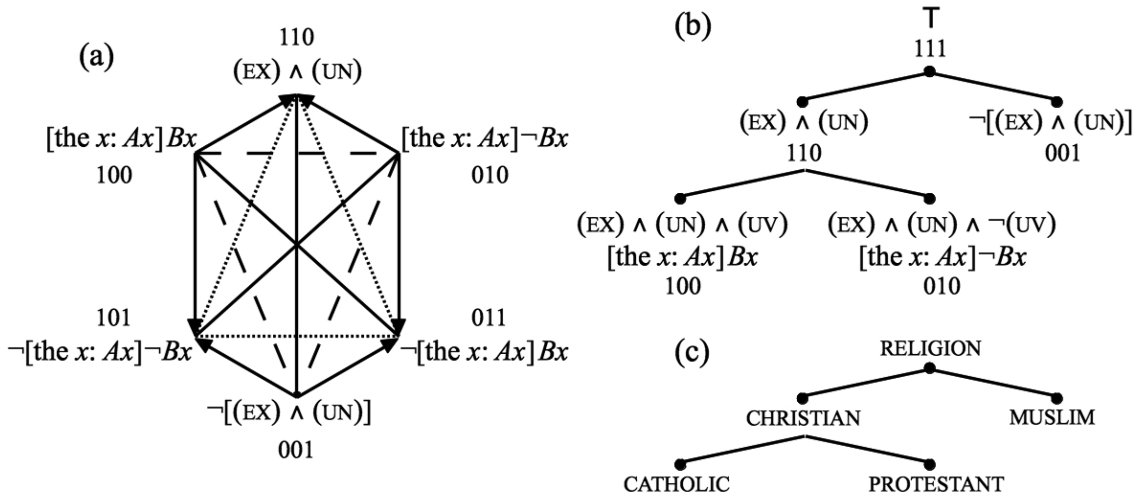
Since this partition has 3 formulas, all definite description formulas in the square and hexagon can be represented by bitstrings of length 3. For example, [the $x\colon Ax]Bx$ is simply α_1, and is thus represented by the bitstring 100; analogously, $\neg$[the $x\colon Ax]\neg Bx$ is FOL-equivalent to $\alpha_1 \vee \alpha_3$, and is thus represented by the bitstring 101. Using bitstrings of length 3, we can represent exactly $2^3 - 2 = 6$ contingent formulas, which are exactly the ones appearing in the JSB hexagon for definite descriptions; see Figure 6(a).[16]

On a more conceptual level, the partition Π^{FOL}_{TDD} can be seen as the result of a process of recursively partitioning and restricting logical space (Seuren, 2014; Seuren and Jaspers, 2014; Jaspers, 2015; Roelandt, 2016). In the first stage, the universe is partitioned into models in which $(\text{EX}) \wedge (\text{UN})$ is true and those in which this formula is false. In the second stage, attention is restricted to the former, and this 'subuniverse' is partitioned into models in which (UV) is true and those in which (UV) is false. This process is summarized in Figure 6(b).

This perspective also sheds new light on the ambiguity of 'the A is not B' that was originally pointed out by Russell. The sentence 'the A is B' unambiguously corresponds

[16]Since it was already shown that the JSB hexagon is the Boolean closure of the square (and of itself), one might argue that the bitstring semantics does not produce any new insights. This objection is misguided, however, for two reasons. First of all, from a strictly *logical* perspective, it is indeed true that if the Boolean closure of a given Aristotelian diagram is already known, then calculating the bitstring semantics for that diagram will not be very useful; however, from a more *conceptual* perspective, that bitstring semantics might still turn out to generate interesting new insights (as will be illustrated in the remainder of this subsection). Secondly, it should be pointed out that the case of the square is quite exceptional, since its Boolean closure is well-known to be a JSB hexagon. For most other Aristotelian diagrams, however, the Boolean closure is not known 'beforehand' and thus needs to be calculated explicitly—and for doing this, it *is* actually very helpful to first calculate the diagram's bitstring semantics (as will be illustrated in Section 5).

Figure 6: (a) The JSB hexagon for definite descriptions with its bitstring semantics, (b) recursive partitioning process for definite descriptions, (c) recursive partitioning process for religions.



to [the x: Ax]Bx (bitstring: 100). The complement (i.e. negation) of this sentence is defined only *relative to a given universe*.[17] If the negation of [the x: Ax]Bx (100) is taken relative to the full universe, we obtain $\neg$[the x: Ax]Bx (011); by contrast, if this negation is taken relative to the subuniverse corresponding to EX $\wedge$ UN (and the third bit position is thus kept constant), we rather obtain [the x: Ax]$\neg Bx$ (010). The two interpretations of 'the A is not B' can thus be seen as the results of negating 'the A is B' relative to two distinct universes.[18] Based on a variety of linguistic and cognitive considerations, Seuren and Jaspers (2014, p. 629) state the following (defeasible) Principle of Complement Selection:

> Natural complement selection is primarily relative to the proximate subuniverse, but there are overriding factors.

For example, given the (extremely simplified) recursive partitioning of religions shown in Figure 6(c), a sentence such as 'John is not a Catholic' is interpreted most naturally as 'John is a Protestant' (negation relative to the subuniverse CHRISTIAN), rather than as 'John is a Protestant or a Muslim' (negation relative to the entire universe RELIGION). However, Seuren and Jaspers's principle is defeasible and can thus be overridden by features such as intonation and additional linguistic material.[19] For example, the sentence 'John is not a Catholic, but I don't know whether he's Protestant or Muslim' sounds perfectly felicitous, which it would not if its first part could only be interpreted as 'John is a Protestant'.

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[17] In set theory, too, the complement of a set A is defined relative to a universe U, viz. $\overline{A} := U \backslash A$.

[18] Note that the distinction between [the x: Ax]$\neg Bx$ and $\neg$[the x: Ax]Bx is usually drawn in strictly *syntactic* terms, relying on the scope of the negation. By viewing these two formulas as the results of negating [the x: Ax]Bx relative to two distinct universes, however, we obtain an alternative, more *semantic* characterization of their (dis)similarities.

[19] Similarly, in Schlenker's Transparancy theory (2007; 2008), the principle *Be Articulate* can be overridden by another pragmatic principle, viz. *Be Brief*.

Moving back to the realm of definite descriptions, Seuren and Jaspers's Principle of Complement Selection thus predicts that the most natural interpretation of 'the A is not B' is [the x: Ax]$\neg Bx$ (negation relative to the subuniverse). This seems intuitively correct, and it is probably the driving force behind Russell's puzzle about the law of excluded middle: on this interpretation, 'the A is B' and 'the A is not B' can indeed be simultaneously false. Furthermore, given the defeasible nature of Seuren and Jaspers's principle, the alternative interpretation of 'the A is not B' is still available, but it needs to be triggered by the appropriate intonation and additional linguistic material (Horn, 1989, pp. 106–107):

(a) *the* largest prime number is not even; in fact, there doesn't *exist* a largest prime
(b) *the* prime divisor of 30 is not even; in fact, 30 has *multiple* prime divisors

Neither (a) nor (b) sound self-contradictory, which they would do if their first parts could only be interpreted as being of the form [the x: Ax]$\neg Bx$. The additional linguistic material in (a) and (b) explicitly states that resp. (EX) and (UN) fail, and thus forces the negation in the first part to be interpreted relative to the entire universe—rather than the subuniverse corresponding to (EX) $\wedge$ (UN)—, which results in the interpretation of the form $\neg$[the x: Ax]Bx.

5 Definite Descriptions and Categorical Statements

In this section we continue our exploration of Aristotelian diagrams for TDD, focusing on those diagrams that describe the subtle interplay between definite description formulas and categorical statements. Subsection 5.1 shows that this interaction gives rise to a Buridan octagon, provides a bitstring semantics for it, and discusses a connection with some recent work on existential import. Next, Subsection 5.2 shows that if one moves from FOL to traditional syllogistics, this Buridan octagon is transformed into another type of Aristotelian octagon. Finally, Subsection 5.3 studies the impact of the uniqueness condition, and discusses a connection with some recent work on public announcement logic.

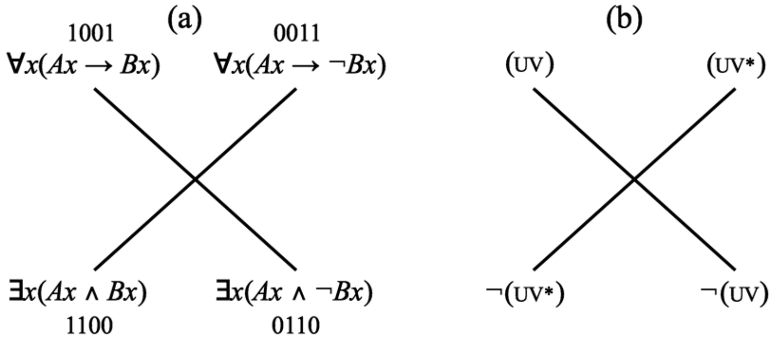
5.1 From Square to Octagon

The four categorical statements from classical syllogistics are 'all As are B', 'some As are B', 'no As are B' and 'some As are not B'; they are traditionally labeled using the vowels A, I, E and O, respectively. These statements can be formalized in the language of first-order logic (FOL) as follows:[20]

$$\begin{array}{llll} \text{A} & \forall x(Ax \to Bx), & \text{E} & \forall x(Ax \to \neg Bx), \\ \text{I} & \exists x(Ax \wedge Bx), & \text{O} & \exists x(Ax \wedge \neg Bx). \end{array}$$

[20]There have also been proposals to formalize the A-statement as $\exists x Ax \wedge \forall x(Ax \to Bx)$ (Chatti and Schang, 2013; Read, 2015), in order to explicitly indicate that this statement has *existential import*. I will return to this suggestion at the end of this subsection and in Subsection 5.2.

Figure 7: (a) Degenerate Aristotelian square for the categorical statements in FOL together with its bitstring semantics, (b) reformulation in terms of $(\textsc{uv})$ and $(\textsc{uv}^*)$.



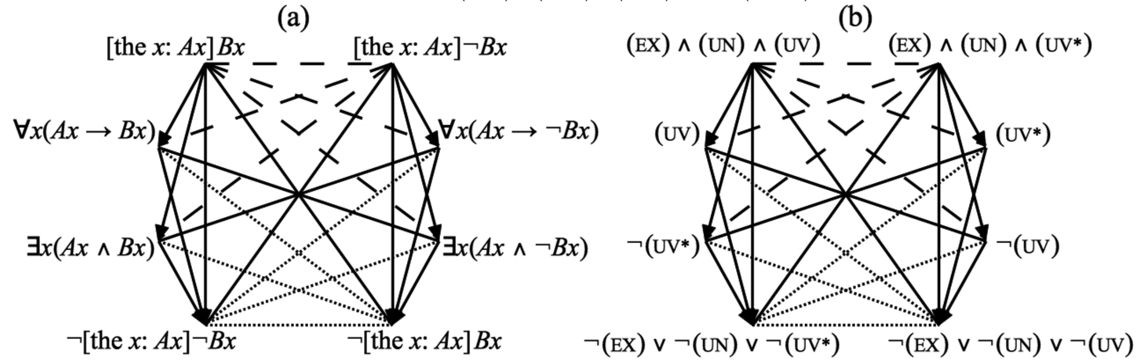
In order to investigate how these categorical statements are related to the definite description formulas from Section 4, one could simply 'add' them to the square in Figure 3(a) and determine which additional Aristotelian relations this gives rise to. In a sense, however, the categorical statements are already present in this definite description square. To see this, recall that [the $x\colon Ax]Bx$ is FOL-equivalent to the conjunction $(\textsc{ex}) \wedge (\textsc{un}) \wedge (\textsc{uv})$, and note that the universality condition $(\textsc{uv})$ is precisely the A-statement. Consequently, the $\neg$[the $x\colon Ax]Bx$ is equivalent to the disjunction $\neg(\textsc{ex}) \vee \neg(\textsc{un}) \vee \neg(\textsc{uv})$, with $\neg(\textsc{uv})$ being equivalent to the O-statement. Completely analogously, [the $x\colon Ax]\neg Bx$ and $\neg$[the $x\colon Ax]\neg Bx$ are equivalent to resp. $(\textsc{ex})\wedge(\textsc{un})\wedge(\textsc{uv}^*)$ and $\neg(\textsc{ex}) \vee \neg(\textsc{un}) \vee \neg(\textsc{uv}^*)$, with the negative universality condition $(\textsc{uv}^*)$ and its negation corresponding to the E- and I-statements, respectively. The four categorical statements are thus already present as conjuncts/disjuncts in the four definite description formulas in Figure 3(a).

It is well-known that in FOL, the categorical statements do not stand in any Aristotelian relation at all, except for the contradictions A/O and E/I (Demey and Smessaert, 2017a, Section 4), and thus yield a 'degenerate' square as shown in Figure 7. The reason for this is precisely that FOL—unlike traditional syllogistics—does not make the assumption of existential import, and thus does not have a subalternation from A to I, etc.[21] As to the interaction between the categorical statements and the definite description formulas, we have on purely Boolean grounds that $(\textsc{ex}) \wedge (\textsc{un}) \wedge (\textsc{uv}) \models_{\mathsf{FOL}} (\textsc{uv})$ but not vice versa, i.e. there is a subalternation in FOL from [the $x\colon Ax]Bx$ to $\forall x(Ax \to Bx)$. Furthermore, it also holds that $(\textsc{ex}) \wedge (\textsc{uv}) \models_{\mathsf{FOL}} \neg(\textsc{uv}^*)$, and thus a fortiori $(\textsc{ex}) \wedge (\textsc{un}) \wedge (\textsc{uv}) \models_{\mathsf{FOL}} \neg(\textsc{uv}^*)$ (but not vice versa), i.e. there is a subalternation in FOL from [the $x\colon Ax]Bx$ to $\exists x(Ax \wedge Bx)$. Continuing along these lines, one can show that the following Aristotelian relations hold in FOL:

- [the $x\colon Ax]Bx$ stands in subalternation to the A- and I-statements, and is contrary to the E- and O-statements,
- [the $x\colon Ax]\neg Bx$ stands in subalternation to the E- and O-statements,

[21] In more colorful language: "In the 20th century [...] the problem of existential import was solved by rejecting the relation of subalternation between the universals and particulars. That rejection results in an austere set of relations, i.e., just the contradictories. The square of opposition became an X of opposition." (Béziau and Payette, 2012, pp. 11–12).

Figure 8: (a) Buridan octagon for definite descriptions and categorical statements, (b) alternative formulation in terms of (EX), (UN), (UV) and (UV*).



and is contrary to the A- and I-statements,

- the A- and I-statements stand in subalternation to $\neg$[the $x\colon Ax]\neg Bx$, and are subcontrary to $\neg$[the $x\colon Ax]Bx$,

- the E- and O-statements stand in subalternation to $\neg$[the $x\colon Ax]Bx$, and are subcontrary to $\neg$[the $x\colon Ax]\neg Bx$.

In sum, the interaction between definite description formulas and categorical statements gives rise to a *Buridan octagon*, as shown in Figure 8. Just like in Subsection 4.1, it should be emphasized that this octagon is entirely defined in FOL, and thus does not depend in any way on the assumption of existential import. In terms of Aristotelian subdiagrams, the octagon contains both the definite description square (Figure 3(a), vertically stretched) and the degenerate square for the categorical statements (Figure 7, horizontally stretched); next, it also has some other subdiagrams that are less relevant for our current purposes.[22]

The bitstring semantics for this Aristotelian octagon can be defined in a highly modular fashion. Recall from Subsection 4.2 that the definite description formulas induce the partition $\Pi^{\mathsf{FOL}}_{TDD} = \{\alpha_1, \alpha_2, \alpha_3\}$ in FOL. Furthermore, it is well-known that in FOL, the categorical statements induce the partition Π^{FOL}_{CAT} containing the following four formulas (Seuren, 2010; Demey and Smessaert, 2017a):[23]

$$\begin{aligned}
\beta_1 &:= \exists x Ax \wedge \forall x(Ax \rightarrow Bx),\\
\beta_2 &:= \exists x(Ax \wedge Bx) \wedge \exists x(Ax \wedge \neg Bx),\\
\beta_3 &:= \exists x Ax \wedge \forall x(Ax \rightarrow \neg Bx),\\
\beta_4 &:= \neg \exists x Ax.
\end{aligned}$$

This partition generates a bitstring semantics for the degenerate square of categorical statements, as shown in Figure 7(a). Since the Buridan octagon consists of the four

[22] Smessaert and Demey (2014b) provide a detailed study of the Aristotelian subdiagrams of the Buridan octagon (and other types of Aristotelian octagons).

[23] In Peirce's logical writings (Hartshorne and Weiss, 1932), we already find what essentially amounts to the partition Π^{FOL}_{CAT} (CP 2.456), immediately followed by the observation that the categorical statements do not yield a classical square in FOL, but rather a degenerate square (CP 2.460).

definite description formulas and the four categorical statements, we know from Section 3 that the partition it induces is $\Pi^{\mathsf{FOL}}_{OCTA} := \Pi^{\mathsf{FOL}}_{TDD} \wedge_{\mathsf{FOL}} \Pi^{\mathsf{FOL}}_{CAT} = \{\alpha \wedge \beta \mid \alpha \in \Pi^{\mathsf{FOL}}_{TDD}, \beta \in \Pi^{\mathsf{FOL}}_{CAT}, \alpha \wedge \beta$ is FOL-consistent$\}$. This partition contains the following formulas:

$$\begin{array}{lcl}
\gamma_1 & := & \exists x \exists y (Ax \wedge Ay \wedge x \neq y) \wedge \forall x (Ax \rightarrow Bx), \\
\gamma_2 & := & \exists x (Ax \wedge Bx) \wedge \exists x (Ax \wedge \neg Bx), \\
\gamma_3 & := & \exists x \exists y (Ax \wedge Ay \wedge x \neq y) \wedge \forall x (Ax \rightarrow \neg Bx), \\
\gamma_4 & := & [\text{the } x\colon Ax] Bx, \\
\gamma_5 & := & [\text{the } x\colon Ax] \neg Bx, \\
\gamma_6 & := & \neg \exists x Ax.
\end{array}$$

It should be noted that $\Pi^{\mathsf{FOL}}_{OCTA}$ is a refinement of both Π^{FOL}_{TDD} and Π^{FOL}_{CAT}. In particular, Π^{FOL}_{TDD} shares γ_4 and γ_5 with $\Pi^{\mathsf{FOL}}_{OCTA}$, while α_3 has been 'split' into the remaining γ_i's (in the sense that $\alpha_3 \equiv_{\mathsf{FOL}} \gamma_1 \vee \gamma_2 \vee \gamma_3 \vee \gamma_6$).[24] Similarly, Π^{FOL}_{CAT} shares γ_2 and γ_6 with $\Pi^{\mathsf{FOL}}_{OCTA}$, while β_1 has been split into γ_1 and γ_4, and β_3 has been split into γ_3 and γ_5 (i.e. $\beta_1 \equiv_{\mathsf{FOL}} \gamma_1 \vee \gamma_4$ and $\beta_3 \equiv_{\mathsf{FOL}} \gamma_3 \vee \gamma_5$). Since $\Pi^{\mathsf{FOL}}_{OCTA}$ has 6 formulas, all formulas in the Buridan octagon can be represented by bitstrings of length 6, as shown in Figure 9(b).

At a conceptual level, the partition $\Pi^{\mathsf{FOL}}_{OCTA}$ is ordered along two semi-independent 'dimensions', viz. (i) the *cardinality* of (the extension of) A, and (ii) the *proportion* of As (if any) that are B. This two-dimensional perspective on $\Pi^{\mathsf{FOL}}_{OCTA}$ is illustrated in Figure 9(a). The reason for calling the cardinality dimension and the proportion dimension *semi*-independent is that the number of available positions along the latter depends on one's position along the former: higher cardinalities of A allow us to make more fine-grained proportionality distinctions.[25]

I will finish this subsection by exploring a connection with some recent work on existential import in syllogistics, such as Seuren (2010), Chatti and Schang (2013) and Read (2015) (also recall Footnote 20). For the sake of concreteness, I will focus on the approach developed by Chatti and Schang (2013) (but also see Footnote 27). Starting from the same first-order formalizations of the categorical statements as we have, Chatti and Schang (2013, pp. 110–111) define for each categorical statement φ a variant $\varphi_{\text{imp!}}$ that explicitly has existential import and a variant $\varphi_{\text{imp?}}$ that explicitly lacks existential import. Formally:[26]

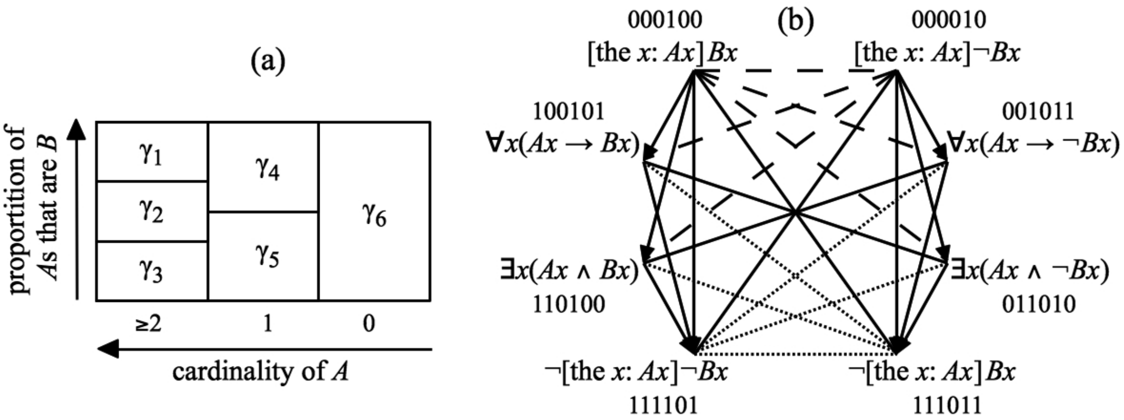
$$\begin{array}{lcl}
\varphi_{\text{imp!}} & := & \exists x Ax \wedge \varphi, \\
\varphi_{\text{imp?}} & := & \neg \exists x Ax \vee \varphi.
\end{array}$$

[24] In particular, note that $\neg(\text{EX}) \equiv_{\mathsf{FOL}} \gamma_6$ and $\neg(\text{UN}) \equiv_{\mathsf{FOL}} \gamma_1 \vee \gamma_2 \vee \gamma_3$, and hence $\alpha_3 \equiv_{\mathsf{FOL}} \neg[(\text{EX}) \wedge (\text{UN})] \equiv_{\mathsf{FOL}} \neg(\text{EX}) \vee \neg(\text{UV}) \equiv_{\mathsf{FOL}} \gamma_6 \vee (\gamma_1 \vee \gamma_2 \vee \gamma_3)$.

[25] In ongoing work with Hans Smessaert and Koen Roelandt, I am investigating whether $\Pi^{\mathsf{FOL}}_{OCTA}$ can also be seen as the result of a cognitively plausible recursive partitioning process—just like Π^{FOL}_{TDD} and the partitioning process illustrated in Figure 6(b). There are at least four plausible candidates: one that is purely based on cardinality, one that is purely based on proportion, and two 'hybrid' partitioning processes, which simultaneously take cardinality and proportion into account, and generate $\Pi^{\mathsf{FOL}}_{OCTA}$ as a refinement of Π^{FOL}_{TDD} or Π^{FOL}_{CAT}, respectively.

[26] Note that $\varphi_{\text{imp?}}$ can be reformulated as $(\text{EX}) \rightarrow \varphi$, and thus expresses the same as φ *conditional upon there being at least one A*. This is analogous to the equivalence between $\neg[\text{the } x\colon Ax]\neg Bx$ and $[(\text{EX}) \wedge (\text{UN})] \rightarrow [\text{the } x\colon Ax]Bx$ discussed in Subsection 4.1, which shows that $\neg[\text{the } x\colon Ax]\neg Bx$ expresses the same as $[\text{the } x\colon Ax]Bx$ *conditional upon there being exactly one A*.

Figure 9: (a) Two-dimensional perspective on the partition $\Pi^{\mathsf{FOL}}_{OCTA}$, (b) the Buridan octagon with its bitstring semantics.



Applying these definitions to the A-, I-, E- and O-statements, we obtain the following 8 formulas:[27]

$$
\begin{array}{llll}
\mathrm{A}_{\mathrm{imp?}} & \equiv_{\mathsf{FOL}} \forall x(Ax \rightarrow Bx) & \equiv_{\mathsf{FOL}} & (\textsc{uv}) \\
\mathrm{I}_{\mathrm{imp!}} & \equiv_{\mathsf{FOL}} \exists x(Ax \wedge Bx) & \equiv_{\mathsf{FOL}} & \neg(\textsc{uv}^*) \\
\mathrm{E}_{\mathrm{imp?}} & \equiv_{\mathsf{FOL}} \forall x(Ax \rightarrow \neg Bx) & \equiv_{\mathsf{FOL}} & (\textsc{uv}^*) \\
\mathrm{O}_{\mathrm{imp!}} & \equiv_{\mathsf{FOL}} \exists x(Ax \wedge \neg Bx) & \equiv_{\mathsf{FOL}} & \neg(\textsc{uv})
\end{array}
$$

$$
\begin{array}{llll}
\mathrm{A}_{\mathrm{imp!}} & \equiv_{\mathsf{FOL}} \exists x Ax \wedge \forall x(Ax \rightarrow Bx) & \equiv_{\mathsf{FOL}} & (\textsc{ex}) \wedge (\textsc{uv}) \\
\mathrm{I}_{\mathrm{imp?}} & \equiv_{\mathsf{FOL}} \neg\exists x Ax \vee \exists x(Ax \wedge Bx) & \equiv_{\mathsf{FOL}} & \neg(\textsc{ex}) \vee \neg(\textsc{uv}^*) \\
\mathrm{E}_{\mathrm{imp!}} & \equiv_{\mathsf{FOL}} \exists x Ax \wedge \forall x(Ax \rightarrow \neg Bx) & \equiv_{\mathsf{FOL}} & (\textsc{ex}) \wedge (\textsc{uv}^*) \\
\mathrm{O}_{\mathrm{imp?}} & \equiv_{\mathsf{FOL}} \neg\exists x Ax \vee \exists x(Ax \wedge \neg Bx) & \equiv_{\mathsf{FOL}} & \neg(\textsc{ex}) \vee \neg(\textsc{uv})
\end{array}
$$

The first four formulas are FOL-equivalent to the 'ordinary' categorical statements, which have already been studied in this subsection. The next four formulas are new, but they are closely related to the four definite description formulas that were studied in Section 4: the only logical/semantic difference is the absence of the $(\textsc{un})$-condition or its negation as a conjunct/disjunct.[28] For example, making use of the third column of the table above, one can easily show that $\mathrm{A}_{\mathrm{imp!}} \wedge (\textsc{un}) \equiv_{\mathsf{FOL}}$ [the $x\colon Ax]Bx$, $\mathrm{I}_{\mathrm{imp?}} \vee \neg(\textsc{un}) \equiv_{\mathsf{FOL}} \neg$[the $x\colon Ax]\neg Bx$, etc.

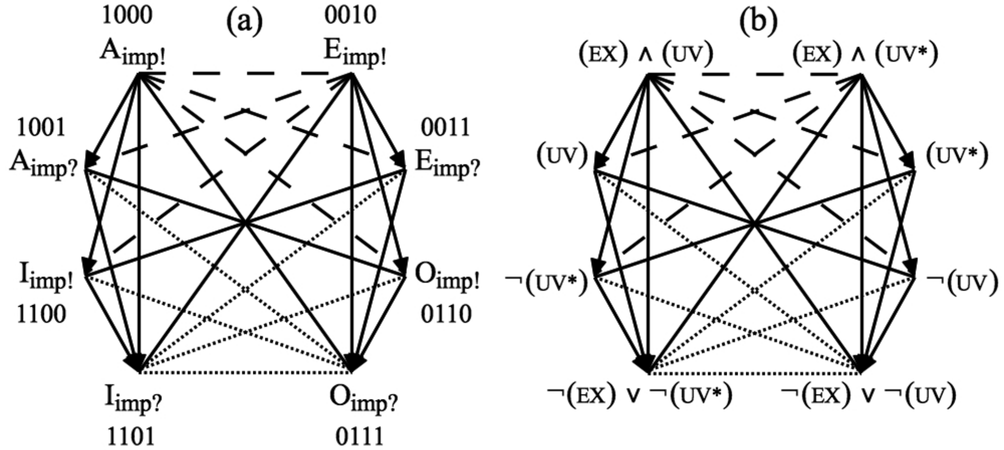
In sum, then, the 8 formulas considered by Chatti and Schang are closely related to the 8 formulas that we have been studying in this subsection.[29] Furthermore, as to

[27]These 8 formulas also figure prominently in Read's (2015) analysis of existential import, but under different names (e.g. $\mathrm{E}_{\mathrm{imp!}}$ and $\mathrm{A}_{\mathrm{imp?}}$ are called A* and E* by Read, respectively). For a full comparison between the two naming conventions, see Chatti and Schang (2013, p. 118).

[28]Recall from Section 2 that the $(\textsc{un})$-condition is sometimes dropped altogether in contemporary quantificational analyses of definite descriptions. Consequently, for such analyses, there is no logical/semantic difference at all between the definite description formulas and Chatti and Schang's categorical statements that explicitly have/lack existential import.

[29]Chatti and Schang explicitly discuss the link to Russell's work in general, writing that their analysis of existential import "owes to Russell many things apart from the symbolism, such as the different scopes of the negation [...] If Russell himself did not give [this analysis], it might be because the problem did not interest him that much or because his aim was not primarily to save Aristotelian logic" (2013, p. 110).

Figure 10: (a) Chatti and Schang's Buridan octagon together with its bitstring semantics, (b) alternative formulation in terms of (EX), (UV) and (UV*).



the Aristotelian relations holding between their 8 formulas, Chatti and Schang (2013, pp. 120–122) show that they constitute a *Buridan octagon* as well, which is shown here in Figure 10.[30] Again, it should be emphasized that although Chatti and Schang are investigating the influence of existential import on the categorical statements, their Buridan octagon is entirely defined in ordinary FOL.

Chatti and Schang themselves do not provide a bitstring semantics for their Buridan octagon, but by applying the technique described in Section 3, one can easily show that their 8 formulas induce the partition $\{\mathrm{A}_{\mathrm{imp!}}, \mathrm{I}_{\mathrm{imp!}} \wedge \mathrm{O}_{\mathrm{imp!}}, \mathrm{E}_{\mathrm{imp!}}, \neg\exists x Ax\}$. Note that this is exactly the same as the partition Π^{FOL}_{CAT} induced by the ordinary categorical statements, which we discussed above. Since this partition has 4 formulas, all formulas in Chatti and Schang's Buridan octagon can be represented by bitstrings of length 4, as shown in Figure 10(a). Furthermore, its Boolean closure contains $2^4 - 2 = 14$ contingent formulas. By contrast, as was shown earlier in this subsection, the Buridan octagon for definite descriptions and categorical statements can be represented by bitstrings of length 6—recall Figure 9(b)—, and its Boolean closure thus contains $2^6 - 2 = 62$ contingent formulas.[31] These two Buridan octagons thus illustrate a phenomenon that is well-known in logical geometry, viz. the fact that two diagrams can have exactly the same *Aristotelian* structure, and yet very different *Boolean* structures (Demey and Smessaert, 2017a).

However, the specific connection with Russell's theory of definite descriptions studied here does not seem to have been noticed before.

[30]Chatti and Schang (2013, p. 122) actually show this Aristotelian diagram as a (3D) *cube*, rather than a (2D) *octagon*. However, this is merely a visual-geometric difference which—as already mentioned at the beginning of Section 3—is irrelevant for our current purposes.

[31]This difference in Boolean structure is also manifested in the diagrams themselves. For example, in Figure 10(a) we have $A_{\mathrm{imp!}} \equiv_{\mathsf{FOL}} A_{\mathrm{imp?}} \wedge I_{\mathrm{imp!}}$ and $I_{\mathrm{imp?}} \equiv_{\mathsf{FOL}} A_{\mathrm{imp?}} \vee I_{\mathrm{imp!}}$ (in terms of the formulas' bitstring representations: $1000 = 1001 \wedge 1100$ and $1101 = 1001 \vee 1100$). By contrast, for the corresponding formulas in Figure 9(b) neither of these equivalences hold: [the x: $Ax]Bx \not\equiv_{\mathsf{FOL}} \forall x(Ax \to Bx) \wedge \exists x(Ax \wedge Bx)$ and $\neg$[the x: $Ax]\neg Bx \not\equiv_{\mathsf{FOL}} \forall x(Ax \to Bx) \vee \exists x(Ax \wedge Bx)$ (in terms of bitstrings: $000100 \neq 100101 \wedge 110100$ and $111101 \neq 100101 \vee 110100$).

5.2 The Assumption of Existential Import

As has been emphasized several times, all Aristotelian diagrams that have been studied so far in this paper are defined in ordinary first-order logic. Nevertheless, througout our investigations we have encountered several issues that seem to suggest a close connection with Aristotelian syllogistics in general, and the question of existential import in particular. For example, we have seen that the (UV)- and (UV^*)-conditions (and their negations) correspond exactly to the (first-order formalizations of the) categorical statements from syllogistics. Furthermore, since the (EX)-condition can be seen as expressing the existential import assumption, Russell's analysis of definite description formulas turns out to be closely related to Chatti and Schang's analysis of the categorical statements as explicitly having/lacking existential import; compare the Buridan octagons in Figures 8 and 10. We will now investigate, therefore, what Russell's TDD—and the Aristotelian diagrams it gives rise to—would look like if it were carried out in Aristotelian syllogistics, rather than ordinary FOL.

As discussed in the previous subsection, Chatti and Schang (2013) deal with existential import by explicitly adding $\exists x Ax$ as a conjunct (or its negation as a disjunct) to the categorical statements. However, one can also think of existential import as a property of the underlying *logical system*, rather than as a property of individual *formulas* (also recall Footnote 20). We therefore introduce the logical system SYL, which is just FOL together with the additional axiom $\exists x Ax$. Naturally, SYL is interpreted on FOL-models $\langle D, I\rangle$ satisfying the additional condition that $I(A) \neq \emptyset$. Recall from Section 3 that the Aristotelian relations are sensitive to the underlying logic; for example, contrariety in FOL is defined as

$$\mathsf{FOL} \models \neg(\varphi \wedge \psi) \quad \text{and} \quad \mathsf{FOL} \not\models \varphi \vee \psi,$$

whereas contrariety in SYL is defined as

$$\mathsf{SYL} \models \neg(\varphi \wedge \psi) \quad \text{and} \quad \mathsf{SYL} \not\models \varphi \vee \psi.$$

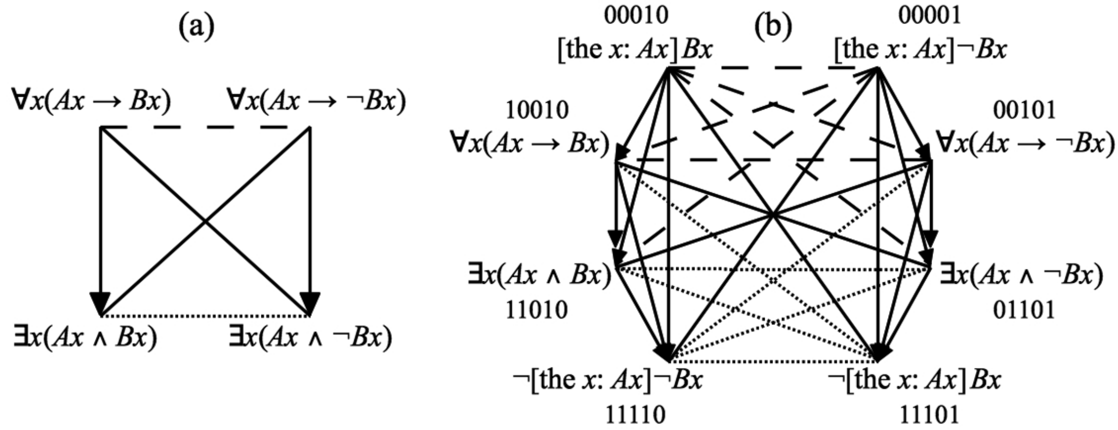
Since FOL and SYL are different logical systems, two formulas can thus be contrary to each other in FOL but not in SYL, or vice versa.[32]

Moving from FOL to SYL has drastic consequences for the Aristotelian relations holding between the categorical statements. As was already mentioned in Subsection 5.1, in FOL these statements yield a degenerate square, as shown in Figure 7. By contrast, in SYL they yield a classical square, as shown in Figure 11(a) (the latter is of course the square that has been taught for centuries in the Aristotelian tradition). For example, while the A- and E-statements are not contrary in FOL (since $\mathsf{FOL} \not\models \neg(\mathrm{A} \wedge \mathrm{E})$), these statements are indeed contrary in SYL (since $\mathsf{SYL} \models \neg(\mathrm{A} \wedge \mathrm{E})$ and $\mathsf{SYL} \models \mathrm{A} \vee \mathrm{E}$).

By contrast, moving from FOL to SYL does not have any influence on the Aristotelian relations holding between the definite description formulas. It was shown in Subsection 4.1 that these formulas yield a classical square in FOL, as shown in Figure 3(a),

[32] Alternatively, one can also continue working in FOL itself, and treat existential import as a *premise*, rather than an *axiom*. For example, the contrariety of φ and ψ would then be defined as $\exists x Ax \models_{\mathsf{FOL}} \neg(\varphi \wedge \psi)$ and $\exists x Ax \not\models_{\mathsf{FOL}} \varphi \vee \psi$. This approach is taken in Nelson (1932). However, in the main text I choose to stick to the 'axiomatic' approach, because of the natural analogy between the quantificational logics FOL/SYL and the modal logics K/D—with the existential import axiom $\exists x Ax$ being the analogue of the consistency axiom $\Diamond\top$ (Chellas, 1980).

Figure 11: (a) Aristotelian square for the categorical statements in SYL, (b) Aristotelian octagon for definite descriptions and categorical statements in SYL.



and one can check that this remains so in SYL. Similarly, the Aristotelian relations between definite description formulas and categorical statements are left unchanged as well. For example, it was shown in Subsection 5.1 that in FOL there are subalternations from $[\text{the } x\colon Ax]Bx$ to the A- and I-statements, and one can check that this remains so in SYL.

In sum, then, the move from FOL to SYL means that we go from the Buridan octagon in Figure 8 to a new Aristotelian octagon, which is shown in Figure 11(b), and will be called a *Lenzen octagon*.[33] The only difference between these two octagons is that the (horizontally stretched) square for the categorical statements has turned from a degenerate into a classical one. This nicely illustrates that the impact of the existential import assumption is restricted to the categorical statements themselves; the definite description formulas—both 'internally' and in their interaction with the categorical statements—are entirely independent of this assumption.

To obtain a bitstring semantics for the new Aristotelian octagon, one might consider 'starting from scratch' and applying the technique described in Section 3 once more, albeit relative to SYL rather than FOL. However, since we are working with the same 8 *formulas* as before (only the *logic* has changed!), there is a more efficient way to proceed. Recall from Subsection 5.1 that in FOL, these 8 formulas induce the partition $\Pi^{\mathsf{FOL}}_{\mathit{OCTA}}$. Although the formula $\gamma_6 = \neg\exists x Ax$ of $\Pi^{\mathsf{FOL}}_{\mathit{OCTA}}$ is consistent in FOL, it is inconsistent in SYL, and thus needs to be dropped from the partition. One can easily check that the 5 other formulas of $\Pi^{\mathsf{FOL}}_{\mathit{OCTA}}$ remain consistent in SYL,[34] and hence we obtain the new partition $\Pi^{\mathsf{SYL}}_{\mathit{OCTA}} := \Pi^{\mathsf{FOL}}_{\mathit{OCTA}} - \{\gamma_6\}$. Since $\Pi^{\mathsf{SYL}}_{\mathit{OCTA}}$ has 5 formulas, all formulas in the Lenzen octagon can be represented by bitstrings of length 5, as shown in Figure 11(b), and furthermore, its Boolean closure has $2^5 - 2 = 30$ contingent formulas. This illustrates how an increase in the logics' axiomatic strength (from FOL to SYL) corresponds to a decrease in the

[33] This type of Aristotelian diagram is much less well-known than the Buridan octagon, but it has already been used by Lenzen (2012) in the context of doxastic-epistemic logic, and by Desclés and Pascu (2012) in their work on the Logic of Determination of Objects.

[34] The formulas γ_4 and γ_5 from $\Pi^{\mathsf{FOL}}_{\mathit{OCTA}}$ can be further simplified in SYL, viz. $\gamma_4 = [\text{the } x\colon Ax]Bx \equiv_{\mathsf{SYL}} \forall x \forall y[(Ax \wedge Ay) \rightarrow (x = y \wedge Bx)]$ and $\gamma_5 = [\text{the } x\colon Ax]\neg Bx \equiv_{\mathsf{SYL}} \forall x \forall y[(Ax \wedge Ay) \rightarrow (x = y \wedge \neg Bx)]$. However, these simplifications are irrelevant for our current purposes.

diagrams' Boolean complexity (from bitstrings of length 6 to bitstrings of length 5).

Finally, it should be noted that since $\Pi^{\mathsf{SYL}}_{OCTA} = \Pi^{\mathsf{FOL}}_{OCTA} - \{\gamma_6\}$, the SYL-bitstrings for the Lenzen octagon in Figure 11(b) are exactly the same as the FOL-bitstrings for the Buridan octagon in Figure 9(b), except for the fact that the sixth bit has systematically been deleted. This deletion (i.e. the SYL-inconsistency of γ_6) thus provides a unified perspective on all the differences and similarities between these two Aristotelian octagons. In particular, in FOL we have that (i) the A- and E-statements both have 1 as their sixth bits, which is the sole reason preventing them from being contrary, (ii) the I- and O-statements both have 0 as their sixth bits, which is the sole reason preventing them from being subcontrary, and (iii) the A- and I-statements have resp. 1 and 0 as their sixth bits, which is the sole reason preventing them from being in subalternation (and similar for the E- and O-statements). By contrast, all other Aristotelian relations are left unchanged when the sixth bit is deleted. For example, [the $x\colon Ax]Bx$ and the E-statement correspond in FOL to the bitstrings 000100 and 001011, which are contrary; after moving from FOL to SYL and deleting the sixth bits, these formulas correspond to the bitstrings 00010 and 00101, which are still contrary.

In conclusion, then, we have seen in this subsection that when a single set of formulas (viz. the 4 definite description formulas and the 4 categorical statements) is analyzed in two logical systems (viz. FOL and SYL), it can give rise to two very different Aristotelian diagrams: the octagons in Figures 9(b) and 11(b) belong to different Aristotelian families, and have different bitstring lengths/Boolean closures. These octagons thus perfectly illustrate the logic-sensitivity of Aristotelian diagrams and the inverse correlation between logical strength and Boolean complexity, two phenomena that are studied in full generality in logical geometry (Demey, 2015; Demey and Smessaert, 2017a).

5.3 The Assumption of Uniqueness

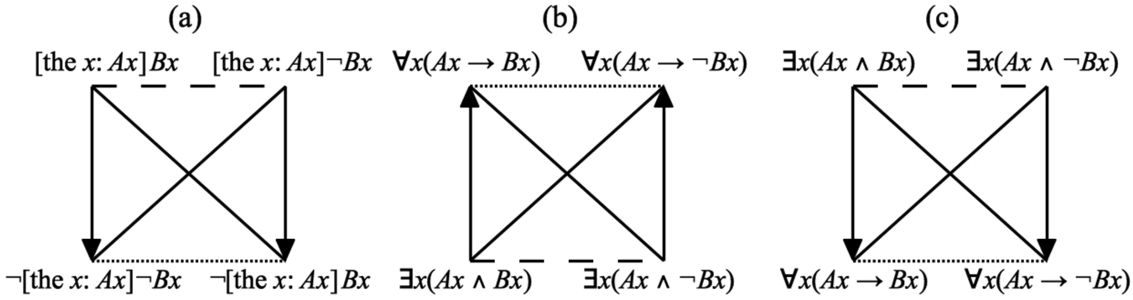
In the previous subsection, we have studied the influence of the existential import assumption on the Buridan octagon for definite description formulas and categorical statements, by adding the (EX)-condition as an axiom to FOL (thereby obtaining the system SYL). However, we have seen in Sections 2 and 4 that (EX) and (UN) fulfill highly complementary roles in Russell's TDD: the former expresses that there is *at least one A*, whereas the latter expresses that there is *at most one A*.[35] In this subsection, we will therefore study how the uniqueness condition influences Russell's TDD and its relation to the categorical statements (as captured in FOL by the Buridan octagon in Figure 8).

We proceed in exactly the same fashion as in the previous subsection, and add $\forall x \forall y[(Ax \wedge Ay) \rightarrow x = y]$ as an additional axiom to FOL. The resulting logical system will be called SYL*.[36] Naturally, SYL* is interpreted on FOL-models $\langle D, I\rangle$ satisfying the additional condition that $|I(A)| \leq 1$. Because of the logic-sensitivity of the Aristotelian relations,

[35] The natural unity of (EX) and (UN) in TDD is also clear from the fact that $\neg$[the $x\colon Ax]\neg Bx$ and $\neg$[the $x\colon Ax]Bx$ are FOL-equivalent to the conditionals $[(\text{EX}) \wedge (\text{UN})] \rightarrow$ [the $x\colon Ax]Bx$ and $[(\text{EX}) \wedge (\text{UN})] \rightarrow$ [the $x\colon Ax]\neg Bx$, respectively, as discussed in Subsection 4.1.

[36] Note that the name SYL* is not meant to suggest any similarity in logical behavior with classical syllogistics; in fact, we will see soon that SYL and SYL* give rise to very different Aristotelian diagrams. Rather, the name is meant to be a reminder that the logics SYL and SYL* are 'dual' to each other, in the sense that they are obtained by adding dual principles to FOL—viz. SYL $=$ FOL $+$ (EX) and SYL* $=$ FOL $+$ (UN).

Figure 12: Aristotelian squares in SYL* for (a) the definite description formulas and (b) the categorical statements; (c) a more standard visual configuration of the Aristotelian square in (b).



it is again to be expected that assuming uniqueness as an axiom will lead to changes in the Aristotelian diagrams.

First of all, it should be noted that moving from FOL to SYL* does not have any influence on the Aristotelian relations holding between the definite description formulas themselves. It was shown in Subsection 4.1 that these formulas yield a classical square in FOL, as shown in Figure 3(a), and one can check that this is also the case in SYL*, as shown in Figure 12(a).

By contrast, moving from FOL to SYL* has a major impact on the Aristotelian relations holding between the categorical statements. It was already mentioned in Subsection 5.1 that in FOL, these statements yield a degenerate square, as shown in Figure 7. One can show, however, that in SYL* they yield a classical square, as shown in Figure 12(b). This square has some remarkable properties: (i) its subalternations run from I and O to resp. A and E (normally it is the other way around), (ii) A and E are subcontraries (normally they are contraries), and (iii) I and O are contraries (normally they are subcontraries). Visually speaking, the square in Figure 12(b) thus seems to have been 'flipped' over its horizontal symmetry axis. A more standard visual configuration of this square is shown in Figure 12(c). We will return to this issue at the end of this subsection.

An equally remarkable situation arises for the Aristotelian relations between the definite description formulas on the one hand and the categorical statements on the other. Consider, for example, [the x: $Ax]Bx$ and the I-statement. In FOL there is a subalternation from the former to the latter, i.e.

(i) FOL $\models$ [the x: $Ax]Bx \rightarrow \exists x(Ax \wedge Bx)$,
(ii) FOL $\not\models \exists x(Ax \wedge Bx) \rightarrow$ [the x: $Ax]Bx$.

Moving from FOL to SYL*, the conditional in (i) remains valid,[37] but the conditional in (ii) now also becomes valid.[38] Consequently, we lose the *subalternation* from [the x: $Ax]Bx$ to $\exists x(Ax \wedge Bx)$; rather, these two formulas turn out to be *equivalent* in SYL*. In exactly

[37]Since SYL* is an extension of FOL, every FOL-validity is also valid in SYL*.

[38]Consider an arbitrary SYL*-model $M = \langle D, I\rangle$ and assume that $M \models \exists x(Ax \wedge Bx)$. Hence there exists an individual $d \in D$ such that $d \in I(A) \cap I(B)$. Since M is a SYL*-model it holds that $|I(A)| \leq 1$, and hence $I(A) = \{d\}$. Furthermore, since we also have $d \in I(B)$, it follows that $I(A) \subseteq I(B)$. Summing up, we have $M \models$ (EX) $\wedge$ (UN) $\wedge$ (UV), i.e. $M \models$ [the x: $Ax]Bx$.

the same way, all 4 definite description formulas and all 4 categorical statements can be shown to be pairwise equivalent in SYL*:

$$\begin{array}{lcccl} [\text{the } x\colon Ax]Bx & \equiv_{\mathsf{SYL}^*} & \mathrm{I} & = & \exists x(Ax \wedge Bx), \\ \neg[\text{the } x\colon Ax]Bx & \equiv_{\mathsf{SYL}^*} & \mathrm{E} & = & \forall x(Ax \rightarrow \neg Bx), \\ [\text{the } x\colon Ax]\neg Bx & \equiv_{\mathsf{SYL}^*} & \mathrm{O} & = & \exists x(Ax \wedge \neg Bx), \\ \neg[\text{the } x\colon Ax]\neg Bx & \equiv_{\mathsf{SYL}^*} & \mathrm{A} & = & \forall x(Ax \rightarrow Bx). \end{array}$$

These pairwise equivalences show that in SYL*, the Aristotelian square for definite descriptions (cf. Figure 12(a)) is simply the same as the Aristotelian square for categorical statements (cf. Figure 12(c)).

These equivalences also change some of the remaining Aristotelian relations between definite description formulas and categorical statements. For example, in FOL it holds that $[\text{the } x\colon Ax]Bx$ is contrary to the E-statement. However, since $[\text{the } x\colon Ax]Bx$ is SYL*-equivalent to I (cf. supra), which is itself contradictory (in FOL as well as SYL*) to E, it follows that $[\text{the } x\colon Ax]Bx$ is no longer contrary, but rather contradictory to E in SYL*. In general, for all definite description formulas φ, ψ and all categorical statements φ', ψ', it holds that if $\varphi \equiv_{\mathsf{SYL}^*} \varphi'$ and $\psi \equiv_{\mathsf{SYL}^*} \psi'$, then the pairs (φ, ψ) and (φ', ψ') stand in the same Aristotelian relation in SYL* (as noted in Section 3, this is an immediate consequence of the fact that the Aristotelian relations are defined up to logical equivalence).

In sum, then, when we move from FOL to SYL*, the Buridan octagon for definite descriptions and categorical statements, as shown in Figure 8, 'collapses' into a classical square in which each vertex represents both a definite description formula and a categorical statement that are SYL*-equivalent to each other, as shown in Figure 13(a).[39] This is thus an even stronger illustration of the logic-sensitivity of Aristotelian diagrams: when a single set of formulas (viz. the 4 definite description formulas together with the 4 categorical statements) is analyzed in two logical systems (viz. FOL and SYL*), it can give rise to two diagrams that not merely belong to different Aristotelian families, but are even of different sizes (viz. the Buridan octagon in Figure 8 and the classical square in Figure 13(a), respectively).

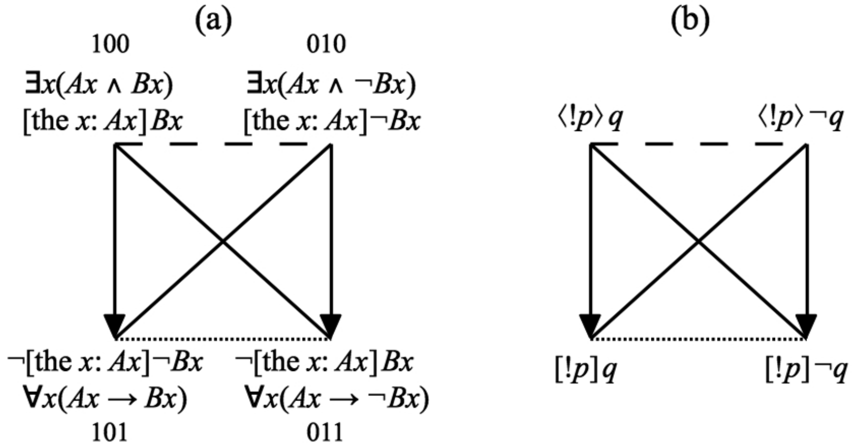
To obtain a bitstring semantics for the collapsed square (*COLL*) in Figure 13(a), one can simply 'start from scratch' and apply the technique described in Section 3 one final time (albeit relative to SYL* rather than FOL), which will yield the partition $\Pi^{\mathsf{SYL}^*}_{COLL} := \{\exists x Ax \wedge \forall x(Ax \rightarrow Bx), \exists x Ax \wedge \forall x(Ax \rightarrow \neg Bx), \neg\exists x Ax\}$. Since this partition has 3 formulas, all formulas in the new square can be represented by bitstrings of length 3, as shown in Figure 13(a), and furthermore, its Boolean closure has $2^3 - 2 = 6$ contingent formulas (i.e. it is a JSB hexagon).[40] However, I will now describe three *alternative* ways to arrive at this bitstring semantics, which offer complementary perspectives on the collapsed square and thus enable a deeper understanding of it.

The *first* approach most closely resembles the one described in the previous subsection. Recalling from Subsection 5.1 that the definite description formulas and categorical statements jointly induce the partition $\Pi^{\mathsf{FOL}}_{OCTA}$ in FOL, we note that the formulas γ_1, γ_2

[39]This square can also be thought of as the result of laying the individual squares in Figures 12(a) and 12(c) on top of each other.

[40]Note, by the way, how an increase in the logics' axiomatic strength (from FOL to SYL*) again brings about a decrease in the diagrams' Boolean complexity (from bitstrings of length 6 to bitstrings of length 3).

Figure 13: (a) Aristotelian square in SYL* for definite descriptions and categorical statements, together with its bitstring semantics, (b) Aristotelian square for public announcement formulas.



and γ_3 are inconsistent in SYL* (since all of them entail that there are at least two As), and thus need to be dropped from this partition. One can easily check that the other formulas of $\Pi^{\mathsf{FOL}}_{OCTA}$ remain consistent in SYL*,[41] and hence we obtain the new partition $\Pi^{\mathsf{FOL}}_{OCTA} - \{\gamma_1, \gamma_2, \gamma_3\}$, which is exactly the partition $\Pi^{\mathsf{SYL}^*}_{COLL}$ already given above. The fact that $\Pi^{\mathsf{SYL}^*}_{COLL} = \Pi^{\mathsf{FOL}}_{OCTA} - \{\gamma_1, \gamma_2, \gamma_3\}$ means that the SYL*-bitstrings for the collapsed square in Figure 13(a) are exactly the same as the FOL-bitstrings for the Buridan octagon in Figure 9(b), except for the fact that the first three bits have systematically been deleted. This offers a unified perspective on all the differences and similarities between these two Aristotelian diagrams. For example, in FOL the formulas $[\text{the } x\colon Ax]Bx$ and $\exists x(Ax \wedge Bx)$ correspond to the bitstrings 000100 and 110100, and hence there is a subalternation from the former to the latter; after moving from FOL to SYL* and deleting the first three bits, these formulas both correspond to the bitstring 100, and are thus equivalent to each other.

As for the *second* perspective on $\Pi^{\mathsf{SYL}^*}_{COLL}$, recall from Subsection 4.2 that the definite description formulas induce the partition $\Pi^{\mathsf{FOL}}_{TDD} = \{\alpha_1, \alpha_2, \alpha_3\}$ in FOL. Since all formulas of Π^{FOL}_{TDD} are not only consistent in FOL, but also in SYL*, the definite description formulas induce the same partition in SYL* as in FOL, viz. Π^{FOL}_{TDD}.[42] Furthermore, since the collapsed square in Figure 13(a) contains precisely the 4 definite description formulas—every definite description formula is accompanied by a SYL*-equivalent categorical statement, but that is irrelevant for the partition induced by those formulas—, it follows that $\Pi^{\mathsf{SYL}^*}_{COLL} = \Pi^{\mathsf{FOL}}_{TDD}$. This means that the SYL*-bitstrings for the collapsed square in Figure 13(a) are exactly the same as the FOL-bitstrings for the definite description square (embedded inside a JSB hexagon) in Figure 6(a).

Finally, the *third* perspective on $\Pi^{\mathsf{SYL}^*}_{COLL}$ can be considered as the dual of the second one. Recall from Subsection 5.1 that the categorical statements induce the partition

[41] The formulas γ_4 and γ_5 from $\Pi^{\mathsf{FOL}}_{OCTA}$ can be further simplified in SYL*, viz. $\gamma_4 = [\text{the } x\colon Ax]Bx \equiv_{\mathsf{SYL}^*} \exists x Ax \wedge \forall x(Ax \to Bx)$, and $\gamma_5 = [\text{the } x\colon Ax]\neg Bx \equiv_{\mathsf{SYL}^*} \exists x Ax \wedge \forall x(Ax \to \neg Bx)$. However, these simplifications are irrelevant for our current purposes.

[42] All formulas of Π^{FOL}_{TDD} can be further simplified in SYL*, viz. $\alpha_1 = [\text{the } x\colon Ax]Bx \equiv_{\mathsf{SYL}^*} \exists x Ax \wedge \forall x(Ax \to Bx)$, $\alpha_2 = [\text{the } x\colon Ax]\neg Bx \equiv_{\mathsf{SYL}^*} \exists x Ax \wedge \forall x(Ax \to \neg Bx)$, and $\alpha_3 = \neg[(\textsc{ex}) \wedge (\textsc{un})] \equiv_{\mathsf{SYL}^*} \neg\exists x Ax$. However, these simplifications are irrelevant for our current purposes.

$\Pi^{\mathsf{FOL}}_{\mathsf{CAT}} = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ in FOL. Although β_2 is consistent in FOL, it is inconsistent in SYL* (since it entails that there are at least two As), and thus needs to be dropped from the partition. One can easily check that the other formulas of Π^{FOL}_{CAT} are consistent in SYL*, and consequently, the partition induced by the categorical statements in SYL* is $\Pi^{\mathsf{FOL}}_{CAT} - \{\beta_2\}$.[43] Furthermore, since the collapsed square in Figure 13(a) contains precisely the 4 categorical statements—every categorical statement is accompanied by a SYL*-equivalent definite description formula, but that is irrelevant for the partition induced by those statements—, it follows that $\Pi^{\mathsf{SYL}^*}_{COLL} = \Pi^{\mathsf{FOL}}_{CAT} - \{\beta_2\}$. This means that the SYL*-bitstrings for the collapsed square in Figure 13(a) are exactly the same as the FOL-bitstrings for the categorical statement square in Figure 7(a), except for the fact that the second bit has systematically been deleted, which offers again a unified perspective on all the differences and similarities between these two squares.

To finish this subsection, I will return to the remarkable fact that the categorical statements constitute a 'flipped' classical square in SYL*—recall Figure 12(b–c). This is essentially due to the fact that the semantics of the categorical statements involves quantifying over $I(A)$, which in SYL*-models is a set containing at most one element. After all, given that $I(A) \leq 1$, the existential claim that *at least one* element from $I(A)$ has some property immediately entails the universal claim that *all* elements from $I(A)$ have that property (but not vice versa, because it might be that $I(A) = \emptyset$). To put this into a broader perspective, I will now discuss how the same phenomenon also arises in a completely different logical system.

Public announcement logic (PAL) is a system of epistemic logic, used to model how agents' knowledge changes under the influence of epistemically relevant events, such as public announcements of formulas (Plaza, 1989; Gerbrandy and Groeneveld, 1997; van Ditmarsch et al., 2007). It makes use of a *dynamic operator* $[!\varphi]$ and its dual $\langle !\varphi \rangle$, which allow us to describe what is the case after the formula φ has been publicly announced. For example, the formula $\neg(K\varphi \vee K\neg\varphi) \wedge [!\varphi]K\varphi$ states that the agent initially does not know whether φ is the case, but after a public announcement of φ, she *does* know that φ is the case. Formally, PAL-formulas are interpreted on pointed Kripke models $(\mathbb{M}, w)$ (with $\mathbb{M} = (W, R, V)$ and $w \in W$); the semantics of the dynamic operators looks as follows:

$$(\mathbb{M}, w) \models [!\varphi]\psi \quad \text{iff} \quad \text{if } (\mathbb{M}, w) \models \varphi \text{ then } (\mathbb{M}^\varphi, w^\varphi) \models \psi,$$
$$(\mathbb{M}, w) \models \langle !\varphi \rangle\psi \quad \text{iff} \quad (\mathbb{M}, w) \models \varphi \text{ and } (\mathbb{M}^\varphi, w^\varphi) \models \psi.$$

These semantic clauses involve moving from the original model $(\mathbb{M}, w)$ to the updated model $(\mathbb{M}^\varphi, w^\varphi)$. Given a pointed Kripke model $(\mathbb{M}, w)$ and a formula φ such that $(\mathbb{M}, w) \models \varphi$, the updated pointed Kripke model $(\mathbb{M}^\varphi, w^\varphi)$ is defined as follows: $w^\varphi := w$ and $\mathbb{M}^\varphi := (W^\varphi, R^\varphi, V^\varphi)$, with $W^\varphi := \{v \in W \mid (\mathbb{M}, v) \models \varphi\}$, $R^\varphi := R \cap (W^\varphi \times W^\varphi)$, and $V^\varphi(p) = V(p) \cap W^\varphi$ for every atom p.

The public announcement formulas can be used to construct a classical Aristotelian square, as shown in Figure 13(b) (Demey, 2012a, 2014).[44] Note that this is also a

[43]Analogously, one can show that the partition induced by the categorical statements in SYL is $\Pi^{\mathsf{FOL}}_{CAT} - \{\beta_4\}$, which, in turn, provides a unified perspective on all the similarities and differences between the degenerate categorical square in FOL (cf. Figure 7) and the classical categorical square in SYL (cf. Figure 11(a)). This is described in detail in Demey and Smessaert (2017a, Section 4).

[44]The bitstring semantics for this square—and for other Aristotelian diagrams for PAL—is defined in Demey and Smessaert (2017a, Section 6).

‘flipped’ square, with the subalternations going from the ‘existential’ ($\Diamond$-like) to the ‘universal’ ($\Box$-like) formulas. This is essentially due to the fact that the model update operation $(\mathbb{M}, w) \mapsto (\mathbb{M}^\varphi, w^\varphi)$ is a *partial function*: either $(\mathbb{M}, w) \models \varphi$, in which case $(\mathbb{M}^\varphi, w^\varphi)$ is defined uniquely (functionality), or $(\mathbb{M}, w) \not\models \varphi$, in which case $(\mathbb{M}^\varphi, w^\varphi)$ is not defined at all (partiality). Partiality and functionality correspond, respectively, to $\mathsf{PAL} \models \langle !\varphi \rangle \psi \rightarrow [!\varphi]\psi$ and $\mathsf{PAL} \not\models [!\varphi]\psi \rightarrow \langle !\varphi \rangle \psi$, and thus jointly justify the subalternations (and implicitly also the other Aristotelian relations) of the square in Figure 13(b).

To make the analogy between (the flipped Aristotelian squares for) SYL^* and PAL fully explicit, note that the latter's dynamic modalities can be seen as quantifying over the set of public announcements. The formulas $[!\varphi]\psi$ and $\langle !\varphi \rangle \psi$ then receive an explicitly universal and existential reading, respectively: $[!\varphi]\psi$ means that ψ holds after *all* public announcements of φ, whereas $\langle !\varphi \rangle \psi$ means that ψ holds after *at least one* public announcement of φ. Furthermore, since the model update operation $(\mathbb{M}, w) \mapsto (\mathbb{M}^\varphi, w^\varphi)$ is a partial function, the set of public announcements of φ will be a set containing at most one element: either $(\mathbb{M}, w) \models \varphi$, in which case there is exactly one public announcement of φ, or $(\mathbb{M}, w) \not\models \varphi$, in which case there is no public announcement of φ at all. In sum, then, the formulas in the PAL-square in Figure 13(b) can also be seen as quantifying over a set of at most one element, just like the categorical statements in the SYL^*-square in Figure 13(a). This explains why both squares are horizontally flipped, with the subalternations going from the existentially to the universally quantified formulas.[45]

6 Conclusion

In this paper I have studied Russell's theory of definite descriptions (TDD) in terms of the Aristotelian diagrams it gives rise to. Russell analyzed sentences of the form ‘the A is B’ in terms of the (EX)-, (UN)- and (UV)-conditions. First, I have argued that each definite description gives rise to four logically distinct formulas, depending on the scope of the negation(s), and shown that these four formulas jointly define a *classical square* (Figure 3). The Boolean closure of this square is a *JSB hexagon* (Figure 4), which can be seen as highlighting the role of the (EX)- and (UN)-conditions in Russell's TDD. Next, I have also extended the definite description square by incorporating the categorical statements, thereby obtaining a *Buridan octagon* (Figure 8), which can be seen as highlighting the role of the (UV)- and (UV*)-conditions in Russell's TDD. Finally, I have studied the exact influence of the (EX)- and (UN)-conditions by moving from ordinary FOL to its extensions SYL and SYL^*, respectively, and showed that this causes the Buridan octagon to turn into a *Lenzen octagon* (Figure 11(b)) or collapse into a *square* (Figure 13(a)).

Along the way, I have emphasized the heuristic value of this diagrammatic analysis, by pointing out several new insights that are relevant for TDD. For example, the need to obtain a complete square of opposition has led us to consider the formula $\neg[\text{the } x\colon Ax]\neg Bx$,

[45] We have drawn an analogy between SYL^* and PAL by showing that the partially functional model update operation $(\mathbb{M}, w) \mapsto (\mathbb{M}^\varphi, w^\varphi)$ from PAL corresponds to a quantification over a set of at most one element, viz. the set of public announcements of φ. Vice versa, the fact that $I(A)$ is a set of at most one element in SYL^*-models $\langle D, I \rangle$ can be used to define a partial function. For any SYL^*-model $M = \langle D, I \rangle$, we let $f(M) := a$ if $a \in I(A)$, and leave $f(M)$ undefined otherwise. It is now easy to check that $M \mapsto f(M)$ is a well-defined partial function.

which does not seem to have been studied before. Similarly, our investigation of the JSB hexagon has led us to consider logical equivalences such as the one between ¬[the $x\colon Ax]Bx$ and $[(\text{EX}) \wedge (\text{UN})] \rightarrow$ [the $x\colon Ax]\neg Bx$, which shed new light on the relations among the various definite description formulas. Finally, while the difference between the formulas [the $x\colon Ax]\neg Bx$ and ¬[the $xAx]Bx$ is usually drawn in a strictly *syntactic* fashion (relying on the notion of scope), I have argued that Seuren and Jaspers's Principle of Complement Selection provides a more *semantic* characterization (as negations of [the $x\colon Ax]Bx$ relative to two different universes).

I have also argued that the Aristotelian diagrams studied in this paper serve as perfect illustrations of various logical phenomena that are studied more systematically in logical geometry. For example, we have seen that a single family of Aristotelian diagrams can have several *Boolean subtypes*: the diagrams in Figures 9(b) and 10(a) are both Buridan octagons, but the former requires bitstrings of length 6, whereas the latter requires bitstrings of length 4. Furthermore, we have encountered several manifestations of the *logic-sensitivity* of Aristotelian diagrams: when a single set of formulas (viz. the 4 definite description formulas and the 4 categorical statements) is analyzed in different logical systems (viz. FOL, SYL and SYL*), it can give rise to Aristotelian diagrams that (i) belong to different Aristotelian families (Buridan octagon vs. Lenzen octagon vs. classical square), (ii) have different Boolean properties (requiring bitstrings of length 6 vs. 5 vs. 3), and (iii) are even of different sizes (octagon vs. square). Finally, by studying logical systems in terms of the Aristotelian diagrams that they give rise to, we introduce a new *layer of abstraction*, which might be helpful for drawing connections between logics that *prima facie* have nothing to do with each other; consider, for example, the 'flipped' Aristotelian squares for SYL* and PAL.

The final conclusion, therefore, is that despite Russell's severe criticisms of Aristotelian logic, there exists a highly fruitful interaction between one of the cornerstones of precisely this logic—viz. the square of opposition and its extensions (as studied today in logical geometry)—and Russell's own quintessential contribution to logical philosophy—viz. his theory of definite descriptions.

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