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The Interaction between Logic and Geometry in Aristotelian Diagrams

Lorenz Demey and Hans Smessaert

Diagrams 2016

1. Introduction
2. Informational and Computational Equivalence
3. Logic versus Geometry in Aristotelian Diagrams
4. Aristotelian Diagrams with 2 PCDs
5. Aristotelian Diagrams with 3 PCDs
6. Conclusion

- Aristotelian diagram: visualization of
  - some formulas/expressions from a given logical/conceptual field
  - the logical relations holding between them
- rich history in philosophical logic (Aristotle); today also used in
  - linguistics (e.g. lexicalization, pragmatics)
  - legal studies (e.g. relations between legal and deontic notions)
  - cognitive science (e.g. fMRI studies on quantifier processing)
  - computer science (e.g. knowledge representation frameworks)
  - etc. . . .

  $\Rightarrow$ lingua franca for interdisciplinary research community
- logical geometry:
  - study Aristotelian diagrams as objects of independent interest
  - abstract-logical aspects
  - visual-geometrical aspects

- our starting point today is the following observation:
  - very often, different authors use vastly different Aristotelian diagrams to visualize one and the same logical structure
  - even after all the logical parameters of a structure have been fixed, there are still several design choices to be made when drawing the diagram
- question: are some of these diagrams 'better' than others?
  - achieve greater positive impact on readers' comprehension of the underlying logical structure
  - cf. the communicative role of Aristotelian diagrams
- goal: propose and illustrate a theory for dealing with this question

- logical system S with Boolean connectives
- two formulas φ and ψ are said to be

S-contradictory	iff	$\mathsf{S} \models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \models \neg(\neg\varphi \wedge \neg\psi)$
S-contrary	iff	$\mathsf{S} \models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \not\models \neg(\neg\varphi \wedge \neg\psi)$
S-subcontrary	iff	$\mathsf{S} \not\models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \models \neg(\neg\varphi \wedge \neg\psi)$
in S-subalternation	iff	$\mathsf{S} \models \varphi \rightarrow \psi$	and	$\mathsf{S} \not\models \psi \rightarrow \varphi$

- two formulas are S-unconnected iff they they do not stand in any Aristotelian relation

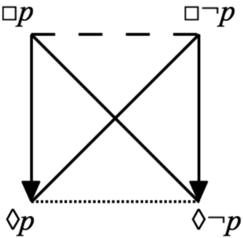
- oldest and most widely known: square of opposition
- throughout history: many other, larger Aristotelian diagrams
  $\Rightarrow$ classification into different families

- a small sample of this classification:
  - classical square (square of opposition)
  - degenerate square (X of opposition)

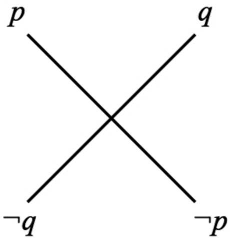
  - Jacoby-Sesmat-Blanché (JSB) hexagon
  - Sherwood-Czeżowski (SC) hexagon
  - unconnectedness-4 (U4) hexagon

  - Béziau octagon
  - Buridan octagon
  - Moretti octagon
  - Keynes-Johnson octagon

classical square

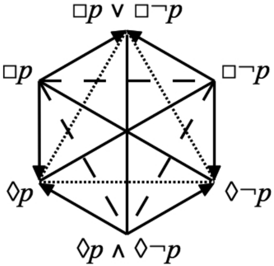


degenerate square

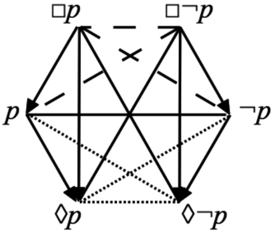


Some families of Aristotelian hexagons

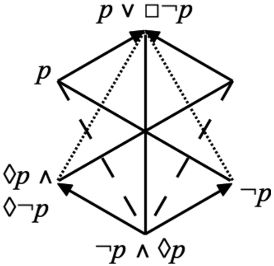
JSB



SC



U4

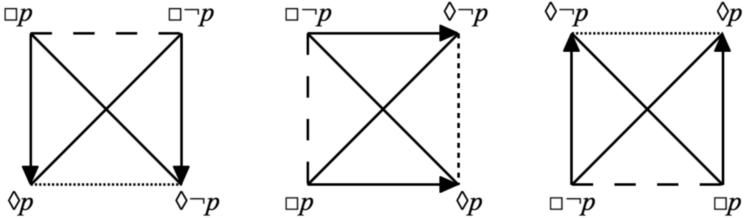


- Aristotelian families are defined in terms of logical properties
  - Aristotelian relations
    - classical square: 2 CD, 1 C, 1 SC, 2 SA
    - degenerate square: 2 CD
  - Boolean structure
    - classical square: Boolean closure is (isomorphic to) $\mathbb{B}_3$
    - degenerate square: Boolean closure is (isomorphic to) $\mathbb{B}_4$

- diagrams belonging to different Aristotelian families are not *informationally equivalent* (Larkin & Simon)
  - visualize different logical structures
  - differences between diagrams $\leftrightsquigarrow$ differences between logical structures

Computational equivalence

- if we focus on diagrams belonging to the same Aristotelian family, we notice that different authors still use vastly different diagrams
- some examples: next slides
- these diagrams are *informationally equivalent*, but not *computationally equivalent* (Larkin & Simon)
  - visualize one and the same logical structure
  - visual differences might influence diagrams' effectiveness (user comprehension of the underlying logical structure)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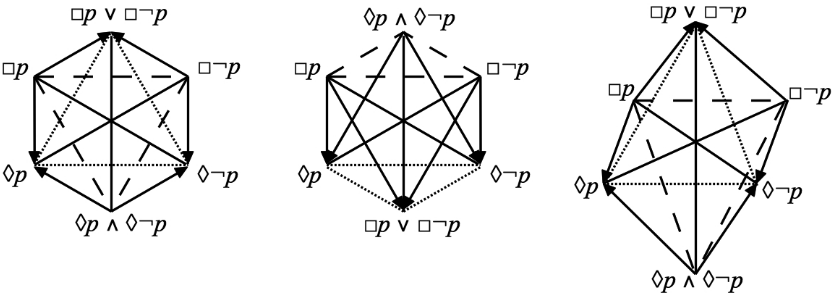
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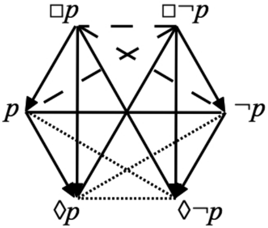

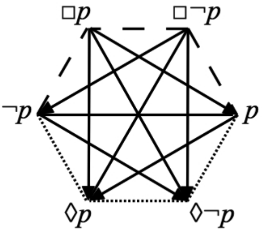

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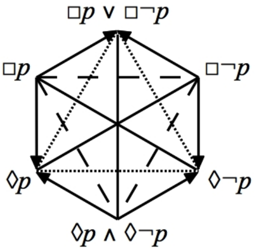
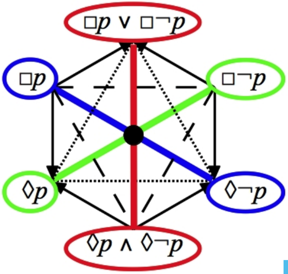
Setup

- two assumptions (satisfied by nearly all diagrams in the literature):
  - the fragment is closed under negation (if $\varphi \in \mathcal{F}$ then $\neg\varphi \in \mathcal{F}$)
  - negation is visualized by means of central symmetry (φ and $\neg\varphi$ occupy diametrically opposed points in the diagram)
- since the fragment is closed under negation, it can be seen
  - as consisting of $2n$ formulas
  - as consisting of n pairs of contradictory formulas (PCDs)

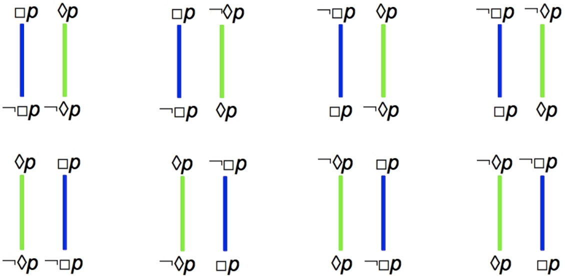




- number of configurations of n PCDs: $2^n \times n!$
  - the n PCDs can be ordered in $n!$ different ways
  - each of the n PCDs has 2 orientations: $(\varphi, \neg\varphi)$ vs. $(\neg\varphi, \varphi)$

- strictly based on the **logical** properties of the fragment
- independent of any concrete visualization

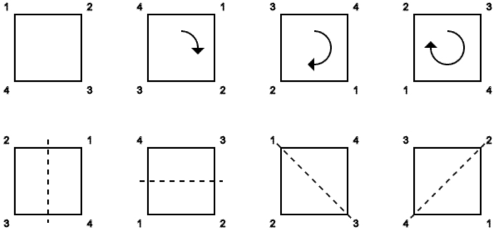
- example: for $n = 2$ PCDs, there are $2^n \times n! = 8$ configurations



- polygon/polyhedron $\mathcal{P}$ to visualize an n-PCD logical fragment
- $\mathcal{P}$ has a symmetry group $\mathcal{S}_{\mathcal{P}}$
  - contains the reflectional and rotational symmetries of $\mathcal{P}$
  - the cardinality $|\mathcal{S}_{\mathcal{P}}|$ measures how 'symmetric' $\mathcal{P}$ is

- strictly based on the **geometrical** properties of the polygon/polyhedron
- independent of the logical structure that is being visualized

- example: a square has 8 reflectional/rotational symmetries, i.e. $|\mathcal{S}_{\mathsf{sq}}| = 8$



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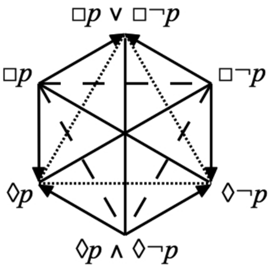
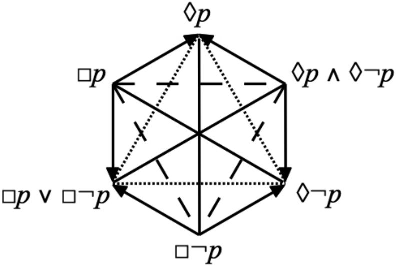
- visualize n-PCD fragment by means of $\mathcal{P}$
  - logical number: $2^n \times n!$
  - geometrical number: $|\mathcal{S}_\mathcal{P}|$

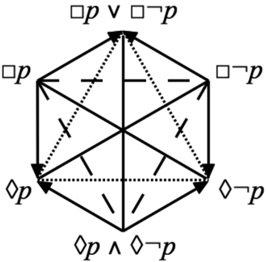
- $2^n \times n! \geq |\mathcal{S}_\mathcal{P}|$ (typically: $>$ instead of $\geq$)
  - every symmetry of $\mathcal{P}$ can be seen as the result of permuting/changing the orientation of the PCDs
  - but typically not vice versa

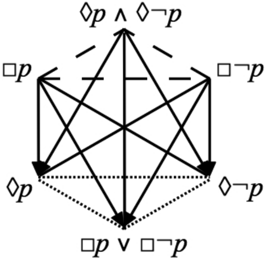
- example
  - reflect the hexagon around the axis defined by $\Box p$ and $\Diamond\neg p$
  - permute the PCDs $(\Diamond p, \Box\neg p)$ and $(\Box p \vee \Box\neg p, \Diamond p \wedge \Diamond\neg p)$





- example
  - change the orientation of the PCD $(\Box p \vee \Box\neg p, \Diamond p \wedge \Diamond\neg p)$
  - no reflectional/rotational symmetry





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- work up to symmetry: $\frac{2^n \times n!}{|\mathcal{S}_\mathcal{P}|}$ fundamental forms
  - diagrams with same fundamental form
    $\Rightarrow$ reflectional/rotational variants of each other
  - diagrams with different fundamental forms:
    $\Rightarrow$ not reflectional/rotational variants of each other

- one n-PCD fragment, two different visualizations $\mathcal{P}$ and $\mathcal{P}'$

  $\mathcal{P}$ is less symmetric than $\mathcal{P}'$

  $\Leftrightarrow |\mathcal{S}_\mathcal{P}| < |\mathcal{S}_{\mathcal{P}'}|$

  $\Leftrightarrow \frac{2^n \times n!}{|\mathcal{S}_\mathcal{P}|} > \frac{2^n \times n!}{|\mathcal{S}_{\mathcal{P}'}|}$

  $\Leftrightarrow$ $\mathcal{P}$ has more fundamental forms than $\mathcal{P}'$

- diagrams $\mathcal{P}$ and $\mathcal{P}'$ for the same n-PCD fragment
  - $\mathcal{P}$ is less symmetric than $\mathcal{P}'$, i.e. has more fundamental forms than $\mathcal{P}'$
  - $\mathcal{P}$ makes some visual distinctions that are not made by $\mathcal{P}'$
- the diagrammatic quality of $\mathcal{P}$ and $\mathcal{P}'$ depends on whether these additional visual distinctions correspond to any logical distinctions in the underlying fragment (Tversky: congruity in diagram design)
- if there are such logical distinctions in the fragment:
  - $\mathcal{P}$ visualizes these logical distinctions (different fundamental forms)
  - $\mathcal{P}'$ collapses these logical distinctions (same fundamental form)
  - $\mathcal{P}$ is better visualization than $\mathcal{P}'$
- if there are no such logical distinctions in the fragment:
  - no need for any visual distinctions either
  - different fundamental forms of $\mathcal{P}$: by-products of its lack of symmetry
  - $\mathcal{P}'$ is better visualization than $\mathcal{P}$

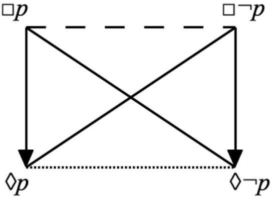
- in general: $\frac{n! \times 2^n}{|\mathcal{S}_{\mathcal{P}}|}$ fundamental forms

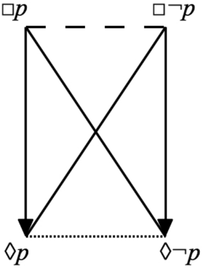
- 2-PCD fragment $\Rightarrow 2! \times 2^2 = 8$ configurations

- some visualizations that have been used in the literature:
  - square: $|\mathcal{S}_{\mathsf{sq}}| = 8$ $\qquad \frac{2! \times 2^2}{|\mathcal{S}_{\mathsf{sq}}|} = \frac{8}{8} = 1$ fundamental form
  - (proper) rectangle: $|\mathcal{S}_{\mathsf{rect}}| = 4$ $\qquad \frac{2! \times 2^2}{|\mathcal{S}_{\mathsf{rect}}|} = \frac{8}{4} = 2$ fundamental forms

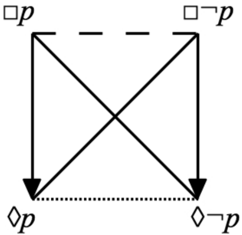
- Aristotelian families of 2-PCD fragments:
  - **classical**
  - degenerate

- 2 fundamental forms
- visual distinction: long vs short edges
  - (sub)contrariety on long edges, subalternation on short edges
  - (sub)contrariety on short edges, subalternation on long edges





- 1 fundamental form
- no visual distinction between long and short edges (all edges are equally long)



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- is there a distinction between (sub)contrariety and subalternation?
- **yes, there is**
  - complementary perspectives on the classical ‘square’ of opposition:
    - as a theory of negation (commentaries on *De Interpretatione*)
    - as a theory of logical consequence (commentaries on *Prior Analytics*)
  - focus on different Aristotelian relations:
    - theory of negation ⇒ focus on (sub)contrariety
    - theory of consequence ⇒ focus on subalternation
  - rectangle does justice to these differences (square would collapse them)
- **no, there isn’t**
  - logical unity of all the Aristotelian relations
    - every (sub)contrariety yields two corresponding subalternations
    - every subalternation yields corresponding contrariety and subcontrariety
  - square does justice to this unity
    (rectangle would introduce artificial differences)

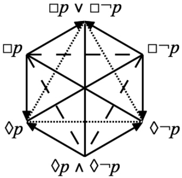
- in general: $\frac{n! \times 2^n}{|\mathcal{S}_\mathcal{P}|}$ fundamental forms

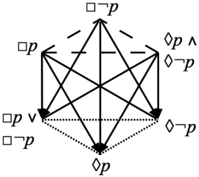
- 3-PCD fragment $\Rightarrow 3! \times 2^3 = 48$ configurations

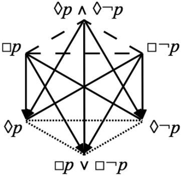
- some visualizations that have been used in the literature:
  - hexagon: $|\mathcal{S}_{\text{hex}}| = 12$ $\qquad \frac{3! \times 2^3}{|\mathcal{S}_{\text{hex}}|} = \frac{48}{12} = 4$ fundamental forms
  - octahedron: $|\mathcal{S}_{\text{octa}}| = 48$ $\qquad \frac{3! \times 2^3}{|\mathcal{S}_{\text{octa}}|} = \frac{48}{48} = 1$ fundamental form

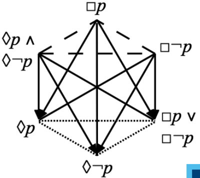
- Aristotelian families of 3-PCD fragments:
  - **Jacoby-Sesmat-Blanché (JSB)**
  - Sherwood-Czeżowski (SC)
  - unconnected-4 (U4)
  - **unconnected-8 (U8)**
  - unconnected-12 (U12)

- 4 fundamental forms
- visual distinction:
  - all three contrariety edges equally long
  - one contrariety edge longer than the other two

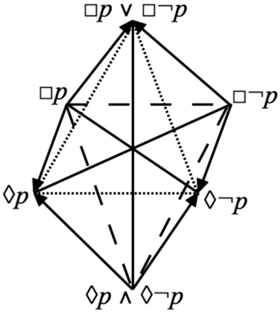








- 1 fundamental form
- no visual distinction between long and short contrariety edges (all contrariety edges are equally long)



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- are there different kinds of contrariety?

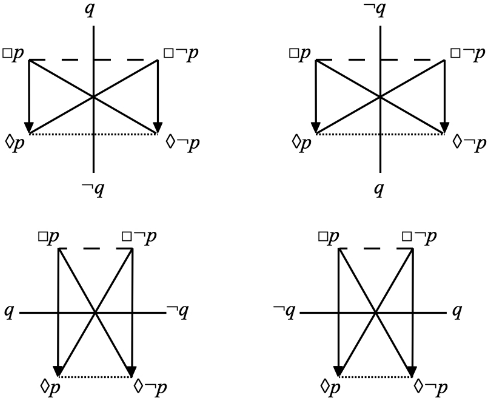
- usually, the contrary formulas are modeled as elements of $\mathbb{B}_3$
  - bitstrings 100, 010 and 001
  - all contrarieties are equally 'strong'

- for linguistic/cognitive reasons, it is sometimes useful to model the contrary formulas as elements of, say, $\mathbb{B}_5$
  - bitstrings 10000, 01110, 00001
  - the contrariety 10000–00001 is 'stronger' than the two other contrarieties

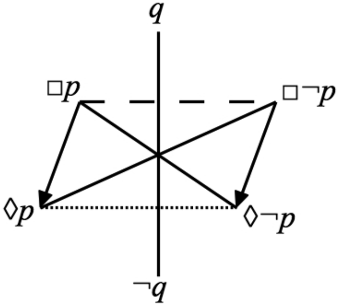
- in the hexagon: edge length $\leftrightsquigarrow$ contrariety strength
- in the octahedron: no distinction possible (collapse)

$\Rightarrow$ hexagon is the preferred visualization

- 4 fundamental forms
- visual distinction:
  - additional PCD parallel to the subalternations
  - additional PCD parallel to the (sub)contrariety



- 1 fundamental form
- no such visual distinction regarding the additional PCD
  (3D polyhedron ⇒ additional PCD pierces through the classical square)



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- there does not seem to be any good logical reason for visualizing the additional PCD as parallel to the (sub)contrariety vs parallel to the subalternations
- in the hexagon visualization, we are forced to make a choice
  - logically unmotivated
  - mere by-product of the lack of symmetry of the hexagon
- in the octahedron visualization, we do not have to make a choice

  $\Rightarrow$ octahedron is the preferred visualization

- systematic approach to informationally equivalent Aristotelian diagrams: logic (PCD structure) vs geometry (symmetry group)
- applied to some Aristotelian families of 2-PCD and 3-PCD fragments

- in general: to visualize an n-PCD fragment, consider a polytope
  - that is centrally symmetric
  - that has $2n$ vertices
  - that has a symmetry group of order $2^n \times n!$

  $\Rightarrow$ *cross-polytope of dimension* n
  (dual of the n-dimensional hypercube)

- diagrammatically ineffective (>3D beyond human visual cognition)
- but theoretically important: first few cases:
  - $n = 2$: 2D cross-polytope: dual of the square: **square**
  - $n = 3$: 3D cross-polytope: dual of the cube: **octahedron**

Thank you!

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