Aristotelian and Duality Relations Beyond the Square of Opposition

Lorenz Demey and Hans Smessaert

Square 2018

Square of opposition:

- represents four propositions, and logical relations between them
- has a long and well-documented history in analytic philosophy, logic and other disciplines

- visually represents the ***Aristotelian relations*** of *contradiction*, *contrariety*, *subcontrariety*, and *subalternation*.
- nearly always also exhibits another type of logical relations, viz. the ***duality relations*** of *internal negation*, *external negation* and *duality*.

- Based on diagrams in the literature, the notions of *Aristotelian square* and *duality square* seem almost co-extensional.
- But, clear conceptual differences between the two!

Logical Geometry:

- systematic study of logical diagrams in general, and Aristotelian diagrams and duality diagrams in particular, in terms of:
  - *cognitive* and *geometric* notions: such as informational vs. computational equivalence, Euclidean distance, vertex-first projections and subdiagrams
  - *logical* issues: diagram informativity, logic-sensitivity, diagram classification and Boolean structure.
- Visual and logical properties of Aristotelian and duality diagrams in isolation are relatively well-understood.

Aim and claims of the paper:

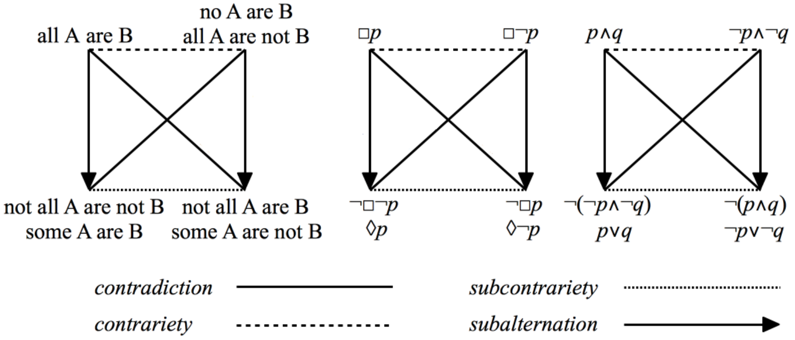
- get clearer picture of interconnections between the two types.
- ***octagons*** are natural extensions/generalizations of the classical square, both from an Aristotelian and duality perspective.
- correspondence is lost on the level of individual relations and diagrams.
- correspondence is maintained on a more abstract level.

1. Introduction
2. Aristotelian and Duality Squares
3. (In)dependence of Aristotelian and Duality Diagrams
4. Octagons for Composed Operator Duality
   - Buridan Octagon in Modal Syllogistics
   - Lenzen Octagon in Modal Logic S4.2
5. Octagons for Generalized Post Duality
   - Keynes-Johnson Octagon in Syllogistics with subject negation
   - Moretti Octagon in Propositional Logic
6. Conclusion

two propositions are:

contradictory	iff	they cannot be true together
(CD)	and	they cannot be false together,
contrary	iff	they cannot be true together
(C)	but	they can be false together,
subcontrary	iff	they can be true together
(SC)	but	they cannot be false together,
in ***subalternation***	iff	the first one entails the second one
(SA)	but	the second one does not entail the first one.

some standard examples:



Contradiction relation:

- most important and informative Aristotelian relation: each proposition φ has a unique contradictory (up to logical equivalence), viz. $\neg\varphi$.
- Almost all Aristotelian diagrams in the literature are closed under contradiction: if the diagram contains φ, then it also contains $\neg\varphi \Rightarrow$ visualized by means of *central symmetry* = diagonals of diagram.
- The propositions in an Aristotelian diagram can naturally be grouped into ***pairs of contradictory propositions*** (PCDs)

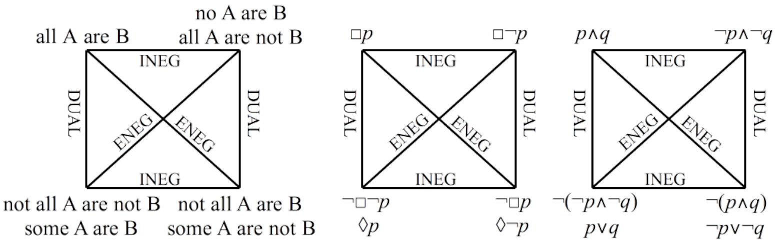
Aristotelian diagrams:

- Shift of perspective: a square does not really consist of 4 'individual' propositions, but rather of 2 PCDs.
- Natural ***extension*** beyond the square, viz. by adding more PCDs:
  - *logically*: from 2 PCDs to 3 PCDs to 4 PCDs to ...
  - *geometrically*: from square to ***hexagon*** to ***octagon*** to ...

- Suppose that two formulas φ and ψ are the results of applying n-ary operators O_φ and O_ψ to the same n propositions $\alpha_1, \ldots, \alpha_n$
- i.e. $\varphi \equiv O_\varphi(\alpha_1, \ldots, \alpha_n)$ and $\psi \equiv O_\psi(\alpha_1, \ldots, \alpha_n)$.
- Then φ and ψ are each other's:

external *negation* (ENEG)	iff	$O_\varphi(\alpha_1, \ldots, \alpha_n) \equiv \neg O_\psi(\alpha_1, \ldots, \alpha_n)$,
internal *negation* (INEG)	iff	$O_\varphi(\alpha_1, \ldots, \alpha_n) \equiv O_\psi(\neg\alpha_1, \ldots, \neg\alpha_n)$,
dual (DUAL)	iff	$O_\varphi(\alpha_1, \ldots, \alpha_n) \equiv \neg O_\psi(\neg\alpha_1, \ldots, \neg\alpha_n)$.

the same standard examples:



- ***functional*** (up to logical equivalence): if $\text{INEG}(\varphi, \psi_1)$ and $\text{INEG}(\varphi, \psi_2)$, then $\psi_1 \equiv \psi_2$, so we write $\psi = \text{INEG}(\varphi)$ instead of $\text{INEG}(\varphi, \psi)$.
- ***symmetrical***: $\text{DUAL}(\varphi, \psi)$ iff $\text{DUAL}(\psi, \varphi)$
- the functions are ***idempotent***: : $\text{ENEG}(\text{ENEG}(\varphi)) = \varphi$
- $\Rightarrow$ define ***identity function*** $\text{ID}(\varphi) := \varphi$ for all φ.

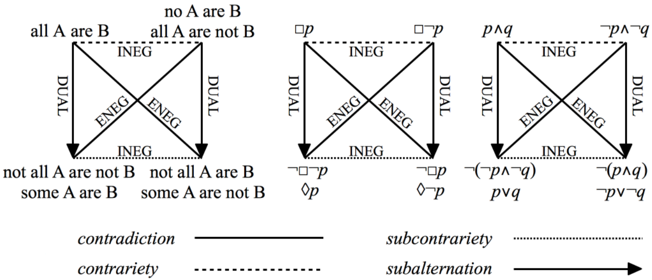
The four duality functions ENEG, INEG, DUAL and ID form a ***Klein 4-group*** under composition ($\circ$) with the following Cayley table:

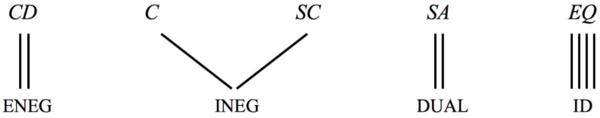
$\circ$	ID	ENEG	INEG	DUAL
ID	ID	ENEG	INEG	DUAL
ENEG	ENEG	ID	DUAL	INEG
INEG	INEG	DUAL	ID	ENEG
DUAL	DUAL	INEG	ENEG	ID

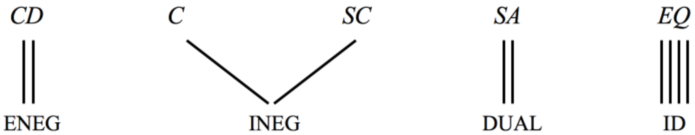
- Klein 4-group is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$: each $\mathbb{Z}_2$ copy governs its own negation: ID $\sim (0,0)$, ENEG $\sim (1,0)$, INEG $\sim (0,1)$, and DUAL $\sim (1,1)$.
- Natural ***extension*** beyond the square of opposition by adding more independent negation positions (i.e. by adding more copies of $\mathbb{Z}_2$):
  - *logically*: from $\mathbb{Z}_2 \times \mathbb{Z}_2$ (2 negation positions $\Rightarrow 2^2 = 4$ duality functions) to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ (3 negation positions $\Rightarrow 2^3 = 8$ duality functions)
  - *geometrically*: from square to ***cube/octagon*** to ...

KU LEUVEN

An ***Aristotelian/duality multigraph (ADM)*** visualizes how many times a specific combination of Aristotelian and duality relation occurs in the square.

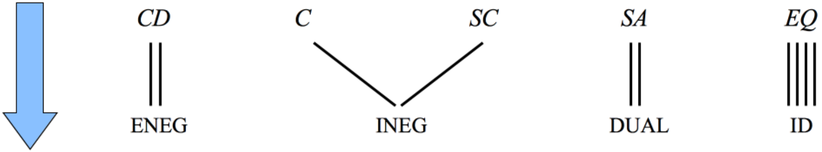






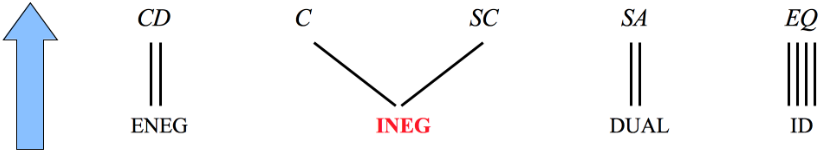
Correspondence between Aristotelian and duality relations is not perfect, but still highly regular.

Aristotelian/Duality Multigraph (ADM)



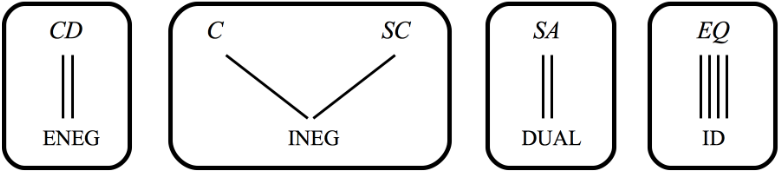
Correspondence between Aristotelian and duality relations is not perfect, but still highly regular.

- each Aristotelian relation corresponds to a unique duality relation.



Correspondence between Aristotelian and duality relations is not perfect, but still highly regular.

- each Aristotelian relation corresponds to a unique duality relation.
- vice versa, duality relations
  - ENEG, DUAL and ID correspond to a unique Aristotelian relation
  - INEG corresponds to two Aristotelian relations.



Correspondence between Aristotelian and duality relations is not perfect, but still highly regular.

- each Aristotelian relation corresponds to a unique duality relation.
- vice versa, duality relations:
  - ENEG, DUAL and ID correspond to a unique Aristotelian relation
  - INEG corresponds to two Aristotelian relations.
- ADM for the square of opposition has 4 ***connected components***, viz. $\{CD, \text{ENEG}\}$, $\{C, SC, \text{INEG}\}$, $\{SA, \text{DUAL}\}$ and $\{EQ, \text{ID}\}$

Close correspondence leads to quasi-identification of two types of squares:

- using Aristotelian terminology to describe duality square (or vice versa)
- viewing one as a generalization of the other

Still some crucial differences:

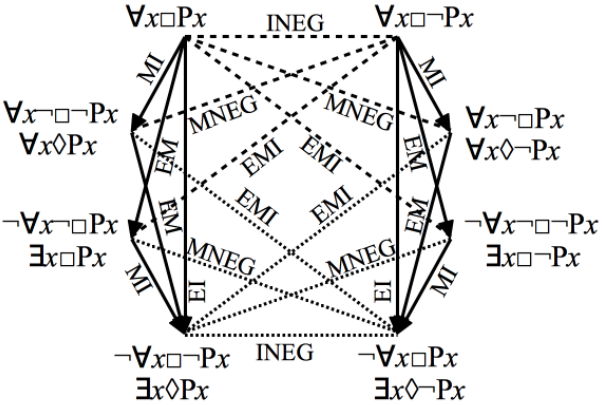
- Duality relations all symmetric $\Leftrightarrow$ Aristotelian *SA* is asymmetric
- Duality relations all functional $\Leftrightarrow$ Aristotelian *C*, *SC* and *SA* are not
- (logic-sensitivity $\Rightarrow$ see full paper)

Most powerful way to argue for independence of Aristotelian and duality diagrams consists in analyzing diagrams *beyond* the square.

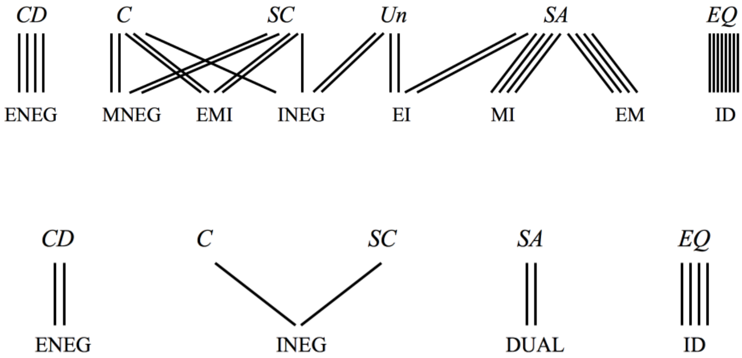
- first attempt — from square to hexagon — was misguided: hexagon is natural extension from Aristotelian but **not** from Duality Perspective.
- ***natural generalisation*** from both perspectives
  $\Rightarrow$ from square ($2 \times 2 = 2^2$) to ***octagon*** ($4 \times 2 = 2^3$)

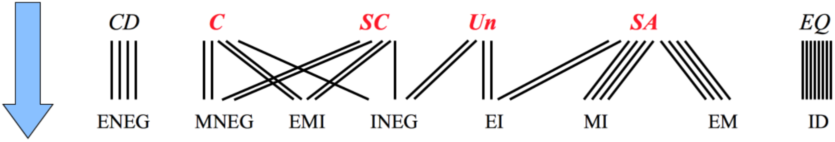
- Suppose that φ is the result of applying an n-ary composed operator $O_1 \circ O_2$ to n propositions $\alpha_1, \ldots, \alpha_n$,
- i.e. $\varphi \equiv (O_1 \circ O_2)(\alpha_1, \ldots, \alpha_n) = O_1(O_2(\alpha_1, \ldots, \alpha_n))$.
- add an extra negation position, viz. *intermediate negation*.
- The proposition $O_1(O_2(\alpha_1, \ldots, \alpha_n))$ has a unique
  - ***external*** *negation* (ENEG): $\neg O_1(O_2(\alpha_1, \ldots, \alpha_n))$,
  - ***intermediate*** *negation* (MNEG): $O_1(\neg O_2(\alpha_1, \ldots, \alpha_n))$,
  - ***internal*** *negation* (INEG): $O_1(O_2(\neg\alpha_1, \ldots, \neg\alpha_n))$.

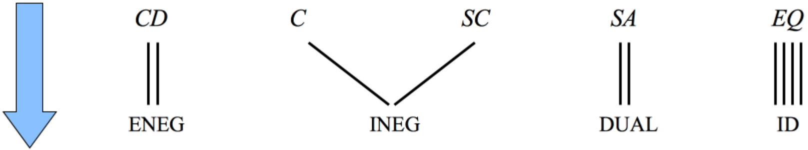
- With 3 independent negation positions, $O_1 \circ O_2$ gives rise to $2^3 = 8$ propositions in total, yielding a much richer duality behavior:
  - ENEG, MNEG, and INEG
  - ENEG $\circ$ INEG (EI), ENEG $\circ$ MNEG (EM), and MNEG $\circ$ INEG (MI).
  - ENEG $\circ$ MNEG $\circ$ INEG (EMI).

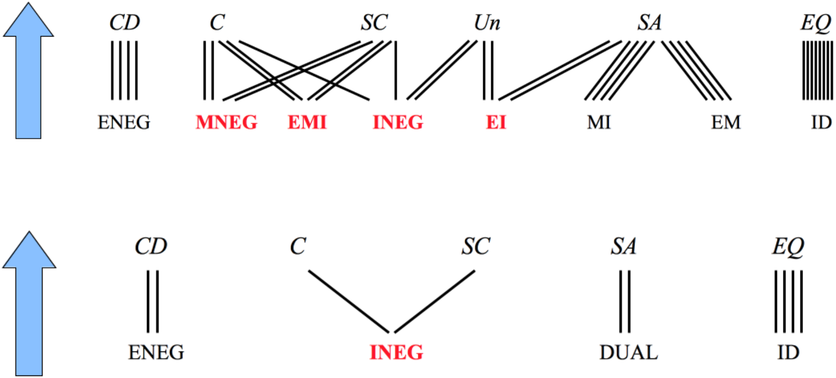


∀x□Px
INEG
∀x□¬Px
MI
MI
∀x¬□¬Px
∀x◊Px
MNEG
MNEG
∀x¬□Px
∀x◊¬Px
EM
EMI
EMI
EM
EM
EMI
EMI
EM
¬∀x¬□Px
∃x□Px
MNEG
MNEG
¬∀x¬□¬Px
∃x□¬Px
MI
EI
EI
MI
¬∀x□¬Px
∃x◊Px
INEG
¬∀x□Px
∃x◊¬Px

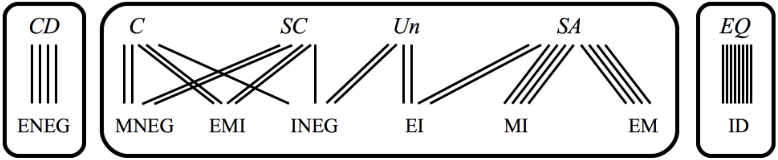

CD
C
SC
Un
SA
EQ
ENEG
MNEG
EMI
INEG
EI
MI
EM
ID
CD
C
SC
SA
EQ
ENEG
INEG
DUAL
ID


CD
C
SC
Un
SA
EQ
ENEG
MNEG
EMI
INEG
EI
MI
EM
ID

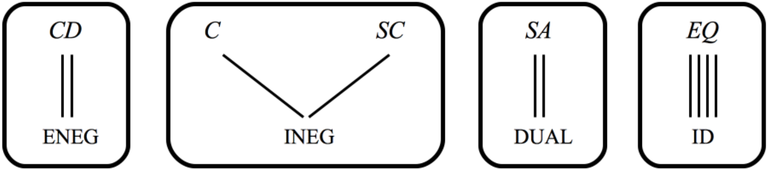

CD
C
SC
SA
EQ
ENEG
INEG
DUAL
ID


CD
C
SC
Un
SA
EQ
ENEG
MNEG
EMI
INEG
EI
MI
EM
ID
CD
C
SC
SA
EQ
ENEG
INEG
DUAL
ID

KU LEUVEN

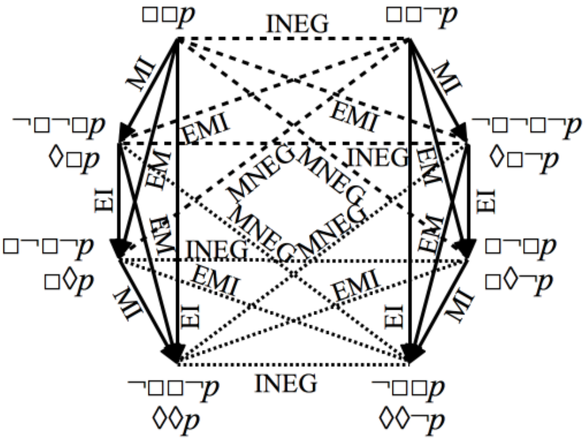


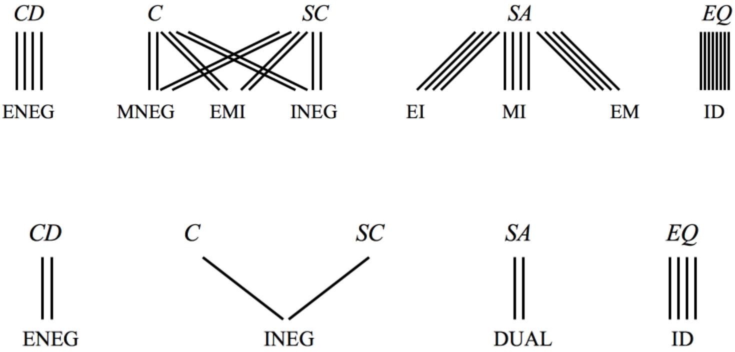
CD
ENEG
C
SC
Un
SA
MNEG
EMI
INEG
EI
MI
EM
EQ
ID



CD
ENEG
C
SC
INEG
SA
DUAL
EQ
ID

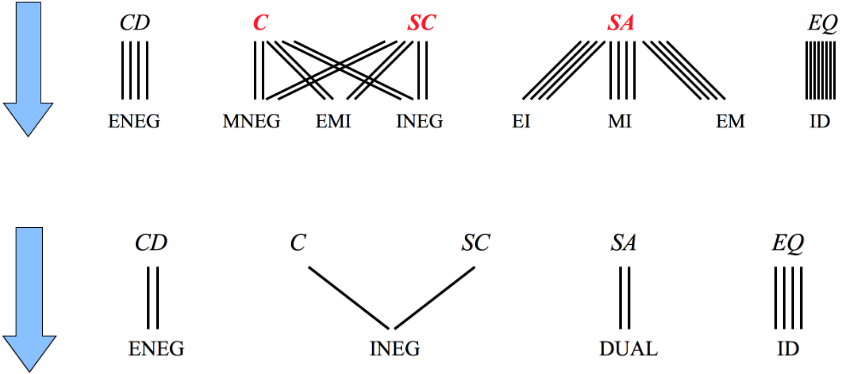
Lenzen Octagon (Modal Logic S4.2)



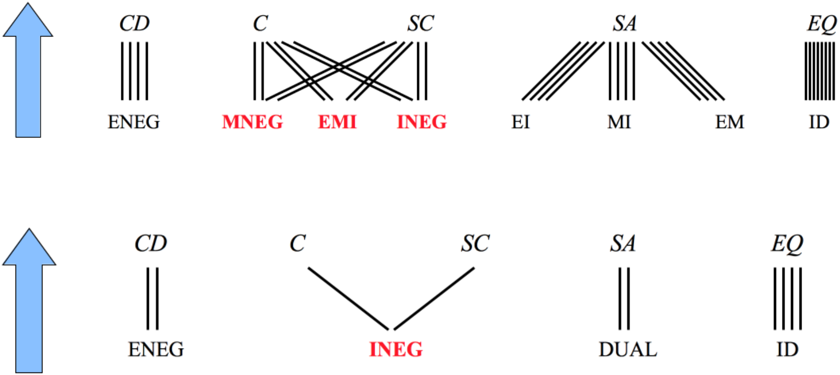


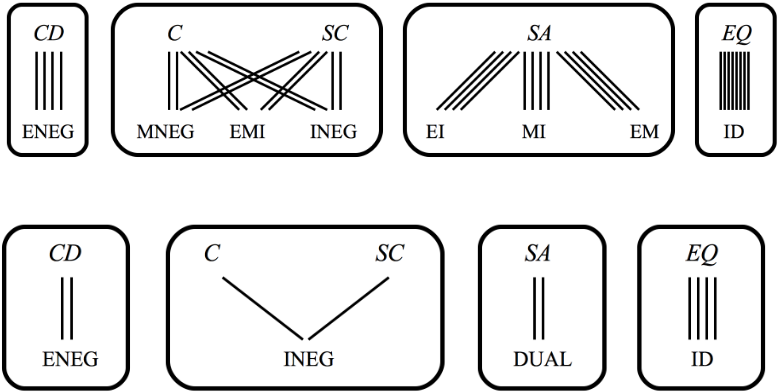
CD
C
SC
SA
EQ
ENEG
MNEG
EMI
INEG
EI
MI
EM
ID
CD
C
SC
SA
EQ
ENEG
INEG
DUAL
ID

Lenzen Octagon (Modal Logic S4.2)



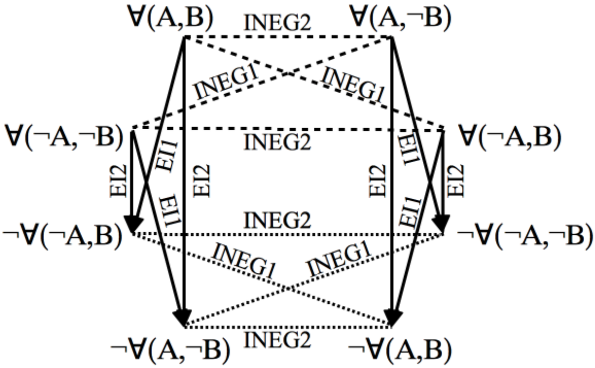
KU LEUVEN


CD
C
SC
SA
EQ
ENEG
MNEG
EMI
INEG
EI
MI
EM
ID
CD
C
SC
SA
EQ
ENEG
INEG
DUAL
ID


CD
ENEG
C
SC
MNEG
EMI
INEG
SA
EI
MI
EM
EQ
ID
CD
ENEG
C
SC
INEG
SA
DUAL
EQ
ID

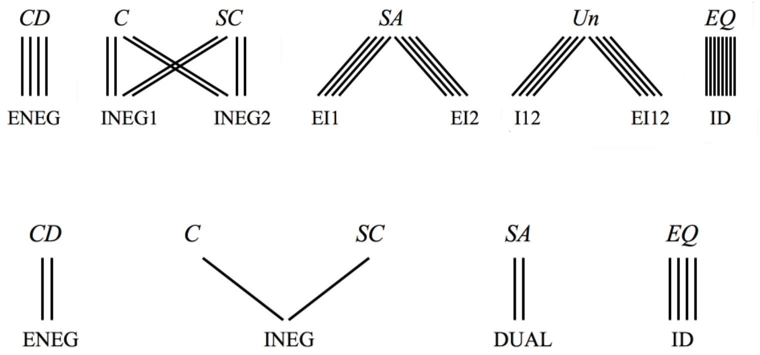
KU LEUVEN

- Classical duality applies internal negation to *all* argument positions, i.e. internal negation of n-ary $O(\alpha_1, \ldots, \alpha_n) \equiv O(\neg\alpha_1, \ldots, \neg\alpha_n)$
- Drop this assumption, and let internal negation apply to each argument position independently: with a binary operator O, we thus have 3 independent negation positions in total.
- The proposition $O(\alpha_1, \alpha_2)$ has a unique:
  - ***external*** *negation* (ENEG): $\neg O(\alpha_1, \alpha_2)$,
  - ***first internal*** *negation* (INEG1): $O(\neg\alpha_1, \alpha_2)$,
  - ***second internal*** *negation* (INEG2): $O(\alpha_1, \neg\alpha_2)$.
- With 3 independent negation positions, $O_1 \circ O_2$ gives rise to $2^3 = 8$ propositions in total, yielding a much richer duality behavior:
  - ENEG, INEG1, and INEG2,
  - ENEG $\circ$ INEG1 (EI1), ENEG $\circ$ INEG2 (EI2), and INEG1 $\circ$ INEG2 (I12),
  - ENEG $\circ$ INEG1 $\circ$ INEG2 (EI12).



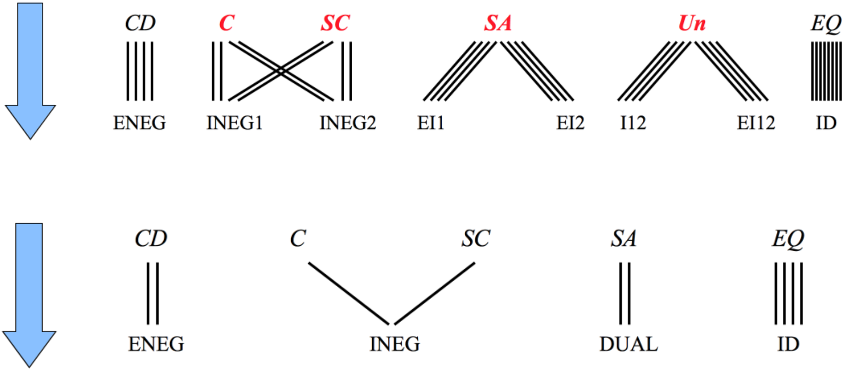
∀(A,B)
INEG2
∀(A,¬B)
INEG1
INEG1
∀(¬A,¬B)
INEG2
∀(¬A,B)
EI1
EI2
EI2
EI1
EI2
EI2
EI1
EI1
¬∀(¬A,B)
INEG2
¬∀(¬A,¬B)
INEG1
INEG1
¬∀(A,¬B)
INEG2
¬∀(A,B)

Keynes-Johnson Octagon (Syllogistics with subject negation)

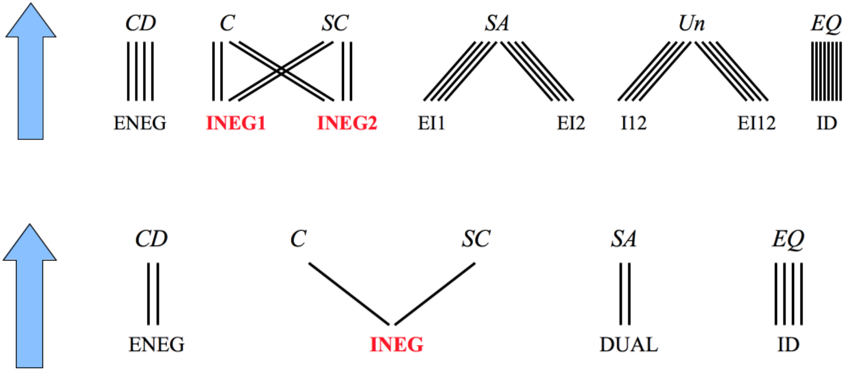


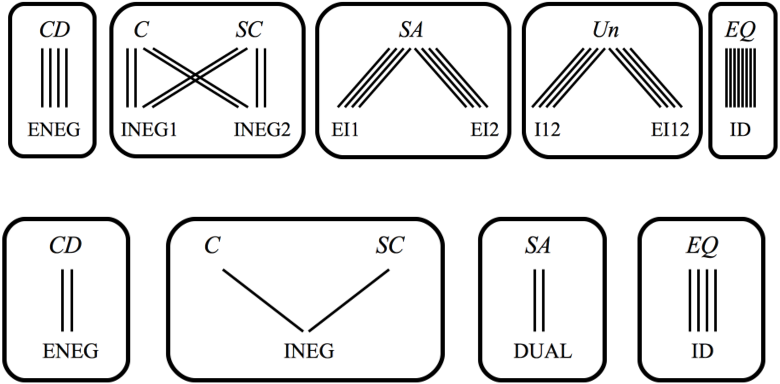
KU LEUVEN

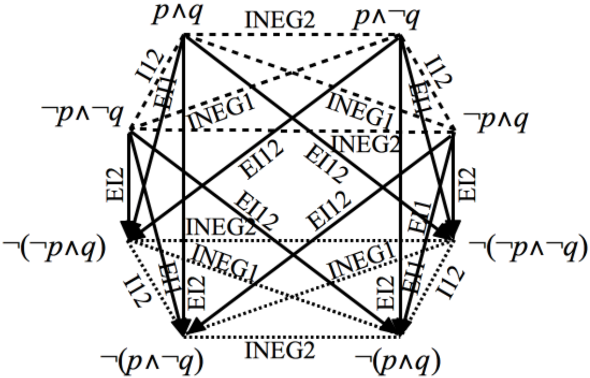
Keynes-Johnson Octagon (Syllogistics with subject negation) 35



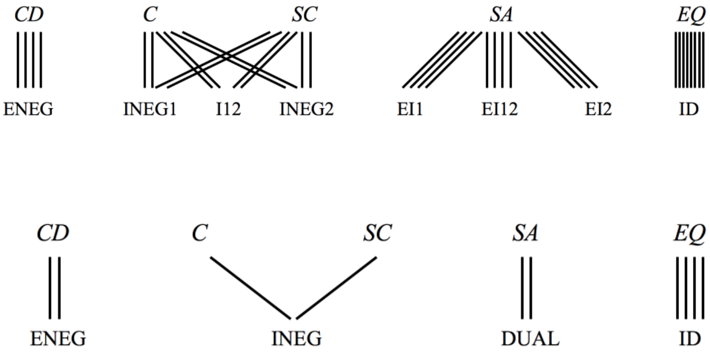
Keynes-Johnson Octagon (Syllogistics with subject negation) 36



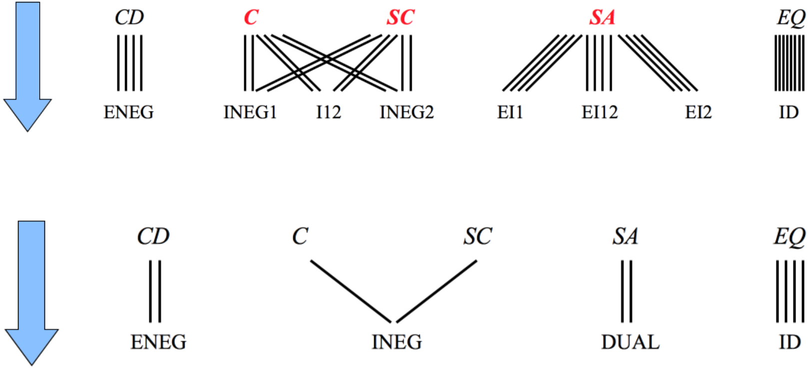

CD
ENEG
C
SC
INEG1
INEG2
SA
EI1
EI2
Un
I12
EI12
EQ
ID
CD
ENEG
C
SC
INEG
SA
DUAL
EQ
ID

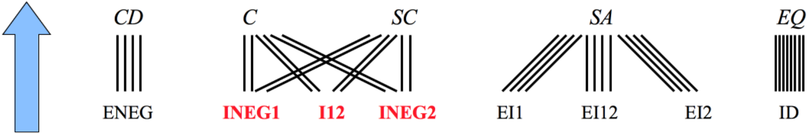


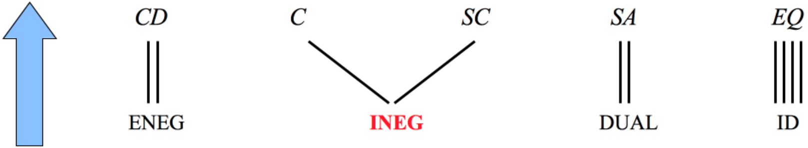
p∧q
INEG2
p∧¬q
¬p∧¬q
¬p∧q
¬(¬p∧q)
¬(¬p∧¬q)
¬(p∧¬q)
¬(p∧q)
II2
EI1
INEG1
EI12
EI2

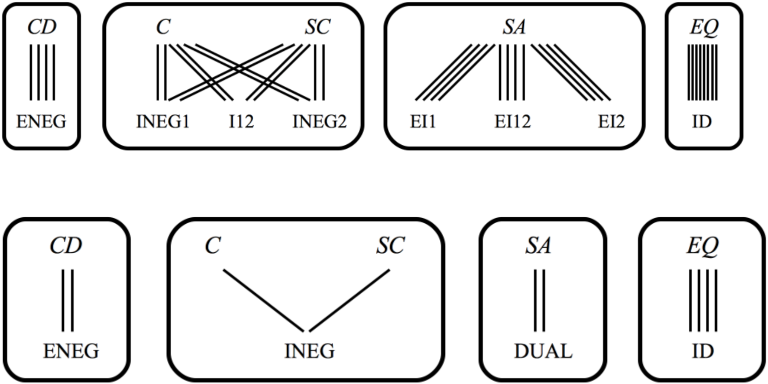


CD
C
SC
SA
EQ
ENEG
INEG1
I12
INEG2
EI1
EI12
EI2
ID
CD
C
SC
SA
EQ
ENEG
INEG
DUAL
ID


CD
C
SC
SA
EQ
ENEG
INEG1
I12
INEG2
EI1
EI12
EI2
ID
CD
C
SC
SA
EQ
ENEG
INEG
DUAL
ID


CD
ENEG
C
SC
INEG1
I12
INEG2
SA
EI1
EI12
EI2
EQ
ID

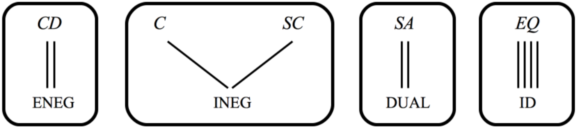

CD
ENEG
C
SC
INEG
SA
DUAL
EQ
ID


CD
ENEG
C
SC
INEG1
I12
INEG2
SA
EI1
EI12
EI2
EQ
ID
CD
ENEG
C
SC
INEG
SA
DUAL
EQ
ID

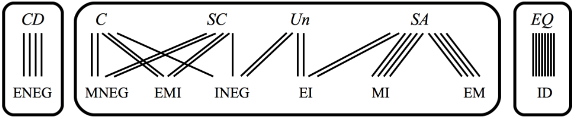
KU LEUVEN

Structure of the talk

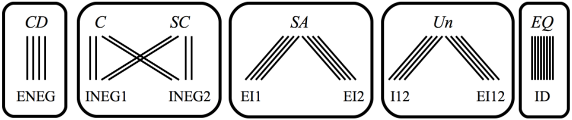
Square = classical duality



Buridan octagon = Composed Operator duality

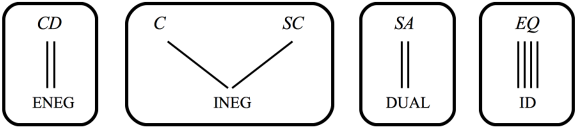


Keynes-Johnson octagon = Generalised Post duality

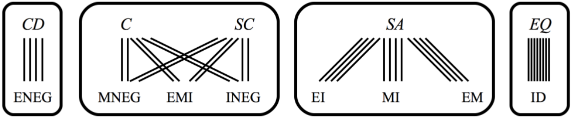


Conclusion

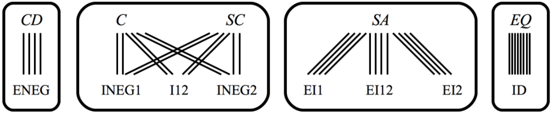
Square = classical duality



Lenzen octagon = Composed Operator duality



Moretti octagon = Generalised Post duality



Thank you!

More info: www.logicalgeometry.org