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Introduction to Logical Geometry

2. Abstract-Logical Properties of Aristotelian Diagrams, Part I

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ESSLLI 2018, Sofia

Structure of the course

1. Basic Concepts and Bitstring Semantics

2. **Abstract-Logical Properties of Aristotelian Diagrams, Part I**
   ☞ **Aristotelian, Opposition, Implication and Duality Relations**

3. Visual-Geometric Properties of Aristotelian Diagrams
   ☞ Informational Equivalence, Symmetry and Distance

4. Abstract-Logical Properties of Aristotelian Diagrams, Part II
   ☞ Boolean Structure and Logic-Sensitivity

5. Case Studies and Philosophical Outlook

- recall the **Aristotelian geometry** $\mathcal{AG}_\mathsf{S} = \{CD_\mathsf{S}, C_\mathsf{S}, SC_\mathsf{S}, SA_\mathsf{S}\}$ (relative to an appropriate logical system S)

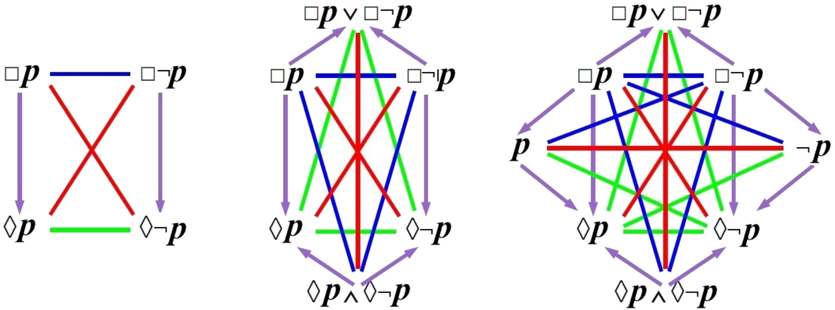
- φ and ψ are said to be

S-contradictory (CD_S)	iff	$\models_\mathsf{S} \neg(\varphi \wedge \psi)$	and	$\models_\mathsf{S} \neg(\neg\varphi \wedge \neg\psi)$
S-contrary (C_S)	iff	$\models_\mathsf{S} \neg(\varphi \wedge \psi)$	and	$\not\models_\mathsf{S} \neg(\neg\varphi \wedge \neg\psi)$
S-subcontrary (SC_S)	iff	$\not\models_\mathsf{S} \neg(\varphi \wedge \psi)$	and	$\models_\mathsf{S} \neg(\neg\varphi \wedge \neg\psi)$
in S-subalternation (SA_S)	iff	$\models_\mathsf{S} \varphi \rightarrow \psi$	and	$\not\models_\mathsf{S} \psi \rightarrow \varphi$

- **Aristotelian square of opposition**: 4 propositions + the Aristotelian relations holding between them

- throughout history: several proposals to extend the square of opposition
  - more propositions, more relations
  - larger and more complex diagrams
  - hexagons, octagons, cubes and other three-dimensional figures

- cf. the motivating examples from lecture 1



- the square and its extensions: various types of hexagons, octagons, etc.
- the extensions are very interesting
  - well-motivated (propositional logic, modal logic S5)
  - throughout history (William of Sherwood, John Buridan, John N. Keynes)
  - interrelations (e.g. JSB hexagon is Boolean closure of classical square)

- yet there is a stunning discrepancy:
  - (nearly) **all** logicians know about the Aristotelian square of opposition
  - (nearly) **no** logicians know about the other Aristotelian diagrams

- our explanation: "the Aristotelian square is very **informative**"
  - this claim sounds intuitive, but is also vague
  - we will provide a precise and well-motivated framework

- recall the Aristotelian geometry $\mathcal{AG}_\mathsf{S}$: φ and ψ are said to be

S-contradictory (CD_S)	iff	$\models_\mathsf{S} \neg(\varphi \wedge \psi)$	and	$\models_\mathsf{S} \neg(\neg\varphi \wedge \neg\psi)$
S-contrary (C_S)	iff	$\models_\mathsf{S} \neg(\varphi \wedge \psi)$	and	$\not\models_\mathsf{S} \neg(\neg\varphi \wedge \neg\psi)$
S-subcontrary (SC_S)	iff	$\not\models_\mathsf{S} \neg(\varphi \wedge \psi)$	and	$\models_\mathsf{S} \neg(\neg\varphi \wedge \neg\psi)$
in S-subalternation (SA_S)	iff	$\models_\mathsf{S} \varphi \to \psi$	and	$\not\models_\mathsf{S} \psi \to \varphi$

- problems with the relations of $\mathcal{AG}_\mathsf{S}$:
  - **not mutually exclusive**: e.g. $\bot$ and p are contrary and subaltern in CPL (lemma: if φ, ψ are contingent, they stand in at most one Arist. relation)
  - **not exhaustive**: e.g. p and $\Diamond p \wedge \Diamond\neg p$ are in no Arist. relation at all in S5 (lemma: if φ is contingent, then φ stands in no Arist. relation to itself)
  - **conceptual confusion**: can be true/false together vs. truth propagation
    - 'together' $\rightsquigarrow$ symmetrical relations (undirected)
    - 'propagation' $\rightsquigarrow$ asymmetrical relations (directed)

- replace subalternation with 'non-contradiction'
- two formulas φ and ψ are said to be

S-contradictory (CD_S)	iff	$\models_S \neg(\varphi \wedge \psi)$	and	$\models_S \neg(\neg\varphi \wedge \neg\psi)$
S-contrary (C_S)	iff	$\models_S \neg(\varphi \wedge \psi)$	and	$\not\models_S \neg(\neg\varphi \wedge \neg\psi)$
S-subcontrary (SC_S)	iff	$\not\models_S \neg(\varphi \wedge \psi)$	and	$\models_S \neg(\neg\varphi \wedge \neg\psi)$
S-non-contradictory (NCD_S)	iff	$\not\models_S \neg(\varphi \wedge \psi)$	and	$\not\models_S \neg(\neg\varphi \wedge \neg\psi)$

- the **opposition geometry** for S: $\mathcal{OG}_S := \{CD_S, C_S, SC_S, NCD_S\}$
- Carnapian state descriptions ('rows 1 and 4 of a truth table'):
  - $\Sigma_1(\varphi, \psi) := \varphi \wedge \psi$
  - $\Sigma_4(\varphi, \psi) := \neg\varphi \wedge \neg\psi$

(note: 'symmetry' between conjuncts of Σ_1 and Σ_4)

- $\mathcal{OG}_S$ is defined of terms $\neg\Sigma_1$ and $\neg\Sigma_4$

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- subalternation: truth propagation ‘from left to right’ $\rightsquigarrow$ left-implication
- vary the ‘direction’ of truth propagation

- two formulas φ and ψ are said to be in

S-bi-implication (BI_S)	iff	$\models_S \varphi \rightarrow \psi$	and	$\models_S \psi \rightarrow \varphi$
S-left-implication (LI_S)	iff	$\models_S \varphi \rightarrow \psi$	and	$\not\models_S \psi \rightarrow \varphi$
S-right-implication (RI_S)	iff	$\not\models_S \varphi \rightarrow \psi$	and	$\models_S \psi \rightarrow \varphi$
S-non-implication (NI_S)	iff	$\not\models_S \varphi \rightarrow \psi$	and	$\not\models_S \psi \rightarrow \varphi$

- the **implication geometry** for S: $\mathcal{IG}_S := \{BI_S, LI_S, RI_S, NI_S\}$

- Carnapian state descriptions (‘rows 2 and 3 of a truth table’):
  - $\Sigma_2(\varphi, \psi) := \varphi \wedge \neg\psi$
  - $\Sigma_3(\varphi, \psi) := \neg\varphi \wedge \psi$

  (note: ‘asymmetry’ between conjuncts of Σ_2 and Σ_3)

- $\mathcal{IG}_S$ is defined of terms $\neg\Sigma_2$ and $\neg\Sigma_3$

- two new geometries: opposition geometry and implication geometry
- together, they solve the problems of the Aristotelian geometry

- the relations of $\mathcal{OG}_S$ are mutually exclusive and jointly exhaustive: each pair of formulas stands in exactly one opposition relation

- the relations of $\mathcal{IG}_S$ are mutually exclusive and jointly exhaustive: each pair of formulas stands in exactly one implication relation

- no longer conceptual confusion:
  - $\mathcal{OG}_S$ is uniformly defined in terms of being able to be true/false together (cf. the symmetrical state descriptions Σ_1 and Σ_4)
  - $\mathcal{IG}_S$ is uniformly defined in terms of truth propagation (cf. the asymmetrical state descriptions Σ_2 and Σ_3)

- clear link with Correia (2012):
  two distinct philosophical traditions in interpreting the square:
  - square as a theory of negation — commentaries on *De Interpretatione*
  - square as a theory of consequence — commentaries on *Prior Analytics*

- terminological remark:
  - 'square of opposition', 'hexagon of opposition', 'cube of opposition'
  - misnomer: exclusive focus on $\mathcal{OG}_\mathsf{S}$, while ignoring $\mathcal{IG}_\mathsf{S}$
  - more appropriate terminology: 'Aristotelian square' etc.
  - concrete examples from the literature:
    - 'square of opposition and equipollence' (John Mikhail, 2007)
    - 'square of implication and opposition' (W. E. Johnson, 1922)
    - 'octagon of implication and opposition' (W. E. Johnson, 1922)

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- opposition and implication geometry are conceptually independent yet there's a clear relationship between them (symmetry breaking):

$$\begin{array}{lcl} CD_{\mathsf{S}}(\varphi,\psi) & \Leftrightarrow & BI_{\mathsf{S}}(\varphi,\neg\psi) \\ C_{\mathsf{S}}(\varphi,\psi) & \Leftrightarrow & LI_{\mathsf{S}}(\varphi,\neg\psi) \\ SC_{\mathsf{S}}(\varphi,\psi) & \Leftrightarrow & RI_{\mathsf{S}}(\varphi,\neg\psi) \\ NCD_{\mathsf{S}}(\varphi,\psi) & \Leftrightarrow & NI_{\mathsf{S}}(\varphi,\neg\psi) \end{array}$$

- both geometries are also internally structured:

$$\begin{array}{lcllcl} CD_{\mathsf{S}}(\varphi,\psi) & \Leftrightarrow & CD_{\mathsf{S}}(\neg\varphi,\neg\psi) & BI_{\mathsf{S}}(\varphi,\psi) & \Leftrightarrow & BI_{\mathsf{S}}(\neg\varphi,\neg\psi) \\ C_{\mathsf{S}}(\varphi,\psi) & \Leftrightarrow & SC_{\mathsf{S}}(\neg\varphi,\neg\psi) & LI_{\mathsf{S}}(\varphi,\psi) & \Leftrightarrow & RI_{\mathsf{S}}(\neg\varphi,\neg\psi) \\ SC_{\mathsf{S}}(\varphi,\psi) & \Leftrightarrow & C_{\mathsf{S}}(\neg\varphi,\neg\psi) & RI_{\mathsf{S}}(\varphi,\psi) & \Leftrightarrow & LI_{\mathsf{S}}(\neg\varphi,\neg\psi) \\ NCD_{\mathsf{S}}(\varphi,\psi) & \Leftrightarrow & NCD_{\mathsf{S}}(\neg\varphi,\neg\psi) & NI_{\mathsf{S}}(\varphi,\psi) & \Leftrightarrow & NI_{\mathsf{S}}(\neg\varphi,\neg\psi) \end{array}$$

- given φ, ψ, we define a binary, truth-functional connective $\circ^{(\varphi,\psi)} = (\circ_1, \circ_2, \circ_3, \circ_4) \in \{0,1\}^4$:
  - φ, ψ stand in exactly one opposition relation

    for $i = 1, 4$, define $\circ_i := \begin{cases} 0 & \text{if } \models_{\mathsf{S}} \neg\Sigma_i(\varphi,\psi) \\ 1 & \text{if } \not\models_{\mathsf{S}} \neg\Sigma_i(\varphi,\psi) \end{cases}$
  - φ, ψ stand in exactly one implication relation

    for $i = 2, 3$, define $\circ_i := \begin{cases} 0 & \text{if } \models_{\mathsf{S}} \neg\Sigma_i(\varphi,\psi) \\ 1 & \text{if } \not\models_{\mathsf{S}} \neg\Sigma_i(\varphi,\psi) \end{cases}$

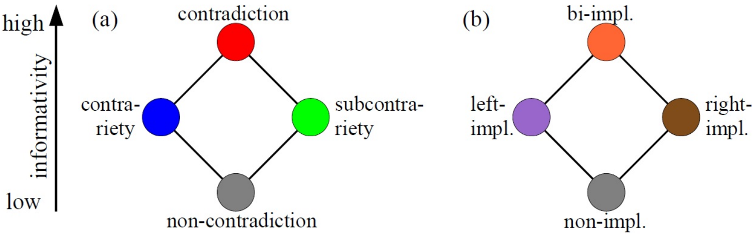
- **theorem**: for all φ, ψ, it holds that $\models \varphi \circ^{(\varphi,\psi)} \psi$
  - e.g.: if $SC_{\mathsf{S}}(\varphi,\psi)$ and $NI_{\mathsf{S}}(\varphi,\psi)$, then $\circ^{(\varphi,\psi)} = (1,1,1,0)$, so $\models_{\mathsf{S}} \varphi \vee \psi$
  - e.g.: if $C_{\mathsf{S}}(\varphi,\psi)$ and $RI_{\mathsf{S}}(\varphi,\psi)$, then $\circ^{(\varphi,\psi)} = (0,1,0,1)$, so $\models_{\mathsf{S}} \neg\psi$

- **theorem**: if φ and ψ are contingent, they can stand in only 7 of the possible 16 ($= 4 \times 4$) combinations of an opp. and an imp. relation

- general idea: the informativity of a statement σ is inversely correlated with the size of its information range $\mathbb{I}(\sigma)$
- **informativity ordering** $\leq_i$: $\sigma \leq_i \tau$ iff $\mathbb{I}(\sigma) \supseteq \mathbb{I}(\tau)$
- we are interested in statements of the form $R_\mathsf{S}(\varphi, \psi)$, with $R_\mathsf{S} \in \mathcal{OG}_\mathsf{S} \cup \mathcal{IG}_\mathsf{S}$
- $\mathbb{I}(R_\mathsf{S}(\varphi, \psi)) := \{\mathbb{M} \in \mathcal{C}_\mathsf{S} \mid \mathbb{M}$ is compatible with $R_\mathsf{S}(\varphi, \psi)\}$
- a model $\mathbb{M}$ of the logic S is said to be **compatible** with $R_\mathsf{S}(\varphi, \psi)$ iff for all $1 \leq i \leq 4 : \Big(R_\mathsf{S}(\varphi, \psi) \;\Rightarrow\; \models_\mathsf{S} \neg\Sigma_i(\varphi, \psi)\Big) \Longrightarrow \mathbb{M} \models \neg\Sigma_i(\varphi, \psi)$
- lift informativity ordering from **statements** $R_\mathsf{S}(\varphi, \psi)$ to **relations** R_S: $R_\mathsf{S} \leq_i^\forall S_\mathsf{S}$ iff $\forall \varphi, \psi : R_\mathsf{S}(\varphi, \psi) \leq_i S_\mathsf{S}(\varphi, \psi)$

- for $1 \leq i \leq 4$, models of type i are those that make $\Sigma_i(\varphi, \psi)$ true
- informativity of the opposition and implication relations:

	models of type		models of type
$CD_S(\varphi,\psi)$	2,3	$BI_S(\varphi,\psi)$	1, 4
$C_S(\varphi,\psi)$	2,3,4	$LI_S(\varphi,\psi)$	1, 3,4
$SC_S(\varphi,\psi)$	1,2,3	$RI_S(\varphi,\psi)$	1,2, 4
$NCD_S(\varphi,\psi)$	1,2,3,4	$NI_S(\varphi,\psi)$	1,2,3,4



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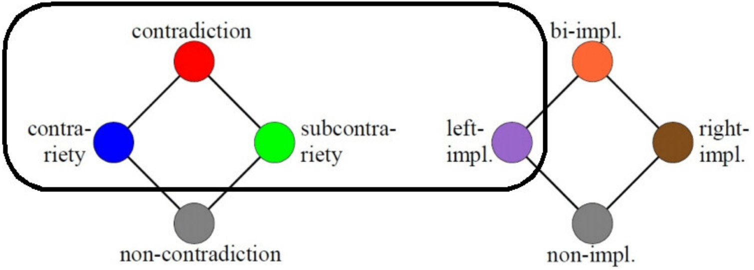
- close match between formal account and intuitions:
  - e.g. CD_S is more informative than C_S
  - if φ is known,
    - announcing $CD_S(\varphi, \psi)$ uniquely determines ψ
    - announcing $C_S(\varphi, \psi)$ does not uniquely determine ψ
- combinatorial results on finite Boolean algebras ($\sim$ bitstrings!)
  - Boolean algebra $\mathbb{B}$ with 2^n formulas, formula of level i:
    - 1 contradictory
    - $2^{n-i} - 1$ contraries and $2^i - 1$ subcontraries
    - $(2^{n-i} - 1)(2^i - 1)$ non-contradictories
  - $1 < 2^{n-i} - 1, 2^i - 1 < (2^{n-i} - 1)(2^i - 1)$ iff $1 < i < n - 1$
- coherence with earlier results:
  - $\mathcal{OG}_S$ and $\mathcal{IG}_S$ yield isomorphic informativity lattices
  - $CD_S(\varphi, \psi) \Leftrightarrow BI_S(\varphi, \neg\psi)$ etc.

The special status of the Aristotelian square

- why is the Aristotelian square special?
- our answer: because it is very informative
  - it is a very informative **diagram** (viz. no unconnectedness)
  - in a very informative **geometry** (viz. the Aristotelian geometry)

Informativity of the Aristotelian geometry, I

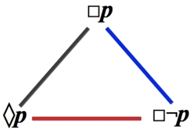
- Aristotelian geometry: hybrid between
  - opposition geometry: contradiction, contrariety, subcontrariety
  - implication geometry: left-implication (subalternation)
- these relations are highly informative (in their geometries)

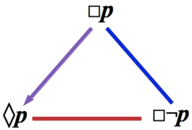


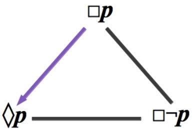
- given any two formulas:
  - they stand in exactly one opposition relation R
  - they stand in exactly one implication relation S

- **theorem**:
  - if R is strictly more informative than S, then R is Aristotelian
  - if S is strictly more informative than R, then S is Aristotelian

- three examples (in S5):
  - $\Box p$ and $\Diamond p$: non-contradiction and **left-implication**
  - $\Box p$ and $\Box\neg p$: **contrariety** and non-implication
  - $\Diamond p$ and $\Box\neg p$: **contradiction** and non-implication

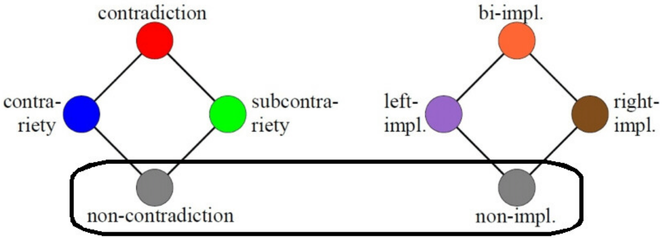






- given any two formulas: one opposition relation, one implication relation
- what if **neither** relation is strictly more informative than the other?

- **theorem**: this can only occur in one case: NCD + NI (unconnectedness)



- Aristotelian gap = information gap
  - no Aristotelian relation at all (recall that $\mathcal{AG}_S$ is not exhaustive)
  - combination of the two least informative relations

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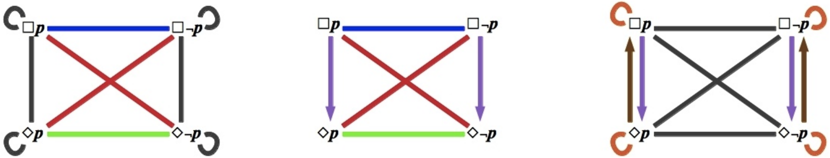
- recall the four-condition characterization of unconnectedness:
  - φ and ψ can be true together — cf. $\Sigma_1(\varphi, \psi)$
  - φ can be true while ψ is false — cf. $\Sigma_2(\varphi, \psi)$
  - φ can be false while ψ is true — cf. $\Sigma_3(\varphi, \psi)$
  - φ and ψ can be false together — cf. $\Sigma_4(\varphi, \psi)$

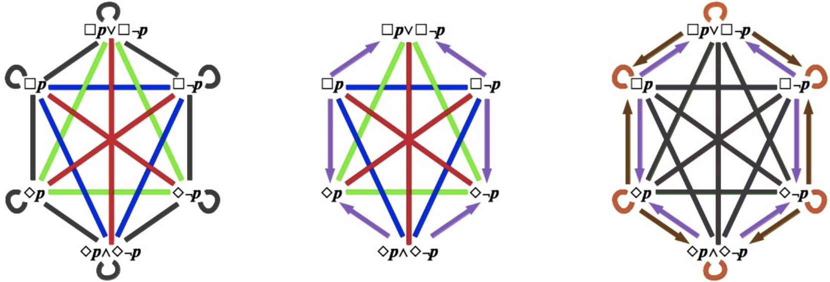
- unconnectedness as the combination of non-contradiction (Σ_1, Σ_4) and non-implication (Σ_2, Σ_3)

- encoding unconnectedness requires **bitstrings of length at least 4**
  - if $\mathbb{B}(\mathcal{F}) \cong \{0,1\}^n$ for $n < 4$, then $\mathcal{F}$ does not contain any pair of unconnected formulas
  - if $\mathcal{F}$ contains at least one pair of unconnected formulas, then $\mathbb{B}(\mathcal{F}) \cong \{0,1\}^n$ for $n \geq 4$

- no unconnectedness in the classical Aristotelian square



- no unconnectedness in the Jacoby-Sesmat-Blanché hexagon



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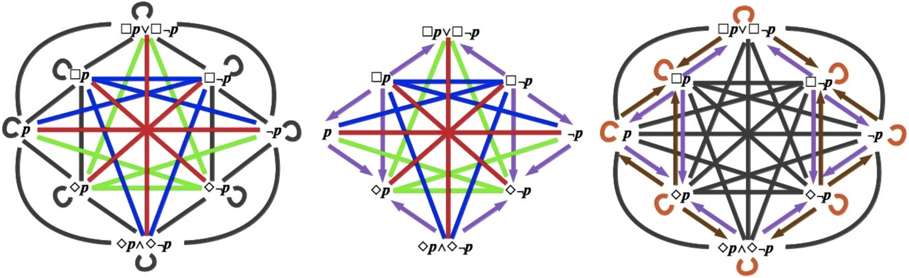
Unconnectedness in some Aristotelian diagrams, II

- unconnectedness in the Béziau octagon
- e.g. p and $\Diamond p \wedge \Diamond\neg p$ are unconnected



Summary: opposition, implication and information

- the Aristotelian **geometry** is hybrid between opposition and implication
- in order to **maximize informativity**

  $\Rightarrow$ applies to all Aristotelian diagrams

- on the level of individual **diagrams**: avoid unconnectedness
- in order to **minimize uninformativity**

  $\Rightarrow$ some Aristotelian diagrams succeed better than others

  - classical square, JSB hexagon, SC hexagon don't have unconnectedness
  - Béziau octagon (and many other diagrams) do have unconnectedness

- Q: what about, say, the JSB hexagon and SC hexagon?
  (equally informative as the square, yet less widely known)
- A: this requires yet another geometry: **duality**

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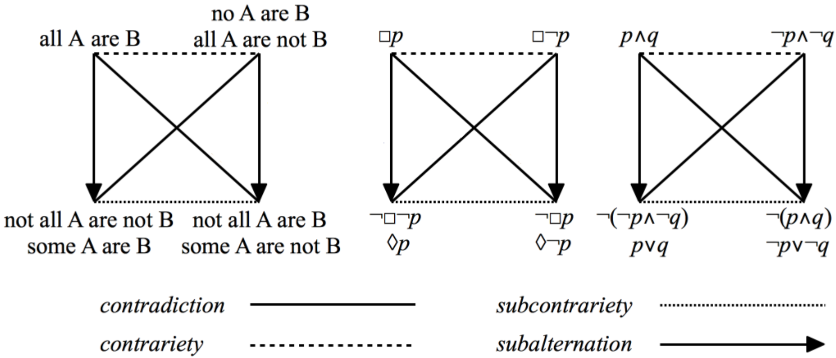
- **square of opposition**:
  - visually represents the **Aristotelian relations** of contradiction, contrariety, subcontrariety and subalternation
  - nearly always also exhibits another type of logical relations, viz. the **duality relations** of internal negation, external negation and duality
- based on the concrete examples found in literature, the notions of **Aristotelian square** and **duality square** seem almost co-extensional
- but: clear conceptual differences between the two!
- the logical and visual properties of Aristotelian and duality diagrams in isolation are relatively well-understood

Aims and claims of this part of the lecture:

- get clearer picture of **interconnections** between the two types of relations
- introduce a new type of diagram to visualise these interconnections: the **Aristotelian/Duality Multigraph** (ADM)
- **octagons** are natural extensions/generalizations of the classical square
  - from an Aristotelian perspective **and**
  - from a duality perspective
- the **correspondence** between Aristotelian and duality relations:
  - is **lost** on the level of individual relations and diagrams
  - is **maintained** on a more abstract level

some standard examples:



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- the **contradiction relation**:
  - most important and informative Aristotelian relation: each proposition φ has a unique contradictory (up to logical equivalence), viz. $\neg\varphi$
  - almost all Aristotelian diagrams in the literature are closed under contradiction: if the diagram contains φ, then it also contains $\neg\varphi$
    $\Rightarrow$ visualized by means of central symmetry ☞ lecture 3
  - the propositions in an Aristotelian diagram can naturally be grouped into **pairs of contradictory propositions** (PCDs)

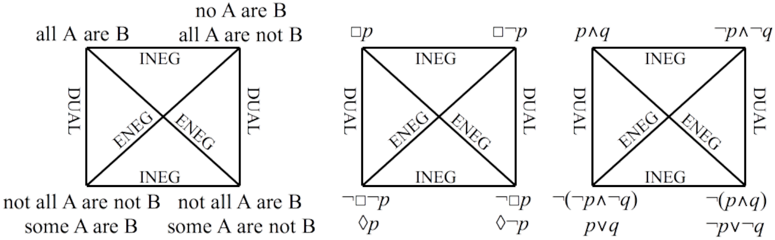
- **Aristotelian diagrams**:
  - remember the shift of perspective:
    - a square does not really consist of 4 individual propositions
    - rather, a square consists of 2 PCDs
  - natural way of extending the square: **adding more PCDs**:
    - logically: from 2 PCDs to 3 PCDs to 4 PCDs to ...
    - geometrically: from square to hexagon to octagon to ...

- suppose that two formulas φ and ψ are the results of applying n-ary operators O_φ and O_ψ to the same n propositions $\alpha_1, \ldots, \alpha_n$
- $\varphi \equiv O_\varphi(\alpha_1, \ldots, \alpha_n)$ and $\psi \equiv O_\psi(\alpha_1, \ldots, \alpha_n)$.
- φ and ψ are said to be each other's

external negation (ENEG)	iff	$O_\varphi(\alpha_1, \ldots, \alpha_n) \equiv \neg O_\psi(\alpha_1, \ldots, \alpha_n)$
internal negation (INEG)	iff	$O_\varphi(\alpha_1, \ldots, \alpha_n) \equiv O_\psi(\neg\alpha_1, \ldots, \neg\alpha_n)$
dual (DUAL)	iff	$O_\varphi(\alpha_1, \ldots, \alpha_n) \equiv \neg O_\psi(\neg\alpha_1, \ldots, \neg\alpha_n)$

Duality relations and squares

the same standard examples:



- the relations are **functional** (up to logical equivalence):
  - e.g. if $\textsc{ineg}(\varphi, \psi_1)$ and $\textsc{ineg}(\varphi, \psi_2)$, then $\psi_1 \equiv \psi_2$
  - we write $\psi = \textsc{ineg}(\varphi)$ instead of $\textsc{ineg}(\varphi, \psi)$
- the relations are **symmetrical**: e.g. $\textsc{dual}(\varphi, \psi)$ iff $\textsc{dual}(\psi, \varphi)$
- the functions are **idempotent**: e.g. $\textsc{eneg}(\textsc{eneg}(\varphi)) = \varphi = \textsc{id}(\varphi)$

- define the **identity function** $\text{ID}(\varphi) := \varphi$
- the four duality functions ID, ENEG, INEG and DUAL form a **Klein 4-group** under composition ($\circ$), with the following Cayley table:

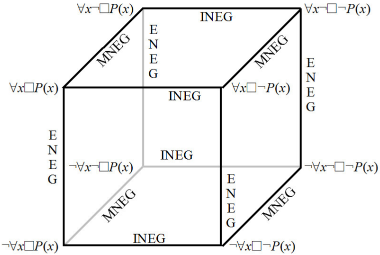
$\circ$	ID	ENEG	INEG	DUAL
ID	ID	ENEG	INEG	DUAL
ENEG	ENEG	ID	DUAL	INEG
INEG	INEG	DUAL	ID	ENEG
DUAL	DUAL	INEG	ENEG	ID

- the Klein 4-group is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$:
  - each copy of $\mathbb{Z}_2$ governs its own negation
  - ID $\sim (0,0)$, ENEG $\sim (1,0)$, INEG $\sim (0,1)$, and DUAL $\sim (1,1)$

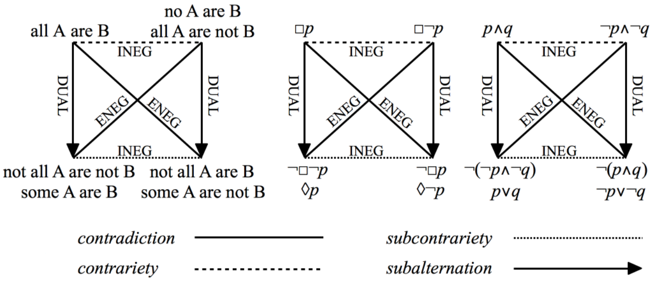
$\circ$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$(0,0)$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$(1,0)$	$(1,0)$	$(0,0)$	$(1,1)$	$(0,1)$
$(0,1)$	$(0,1)$	$(1,1)$	$(0,0)$	$(1,0)$
$(1,1)$	$(1,1)$	$(0,1)$	$(1,0)$	$(0,0)$

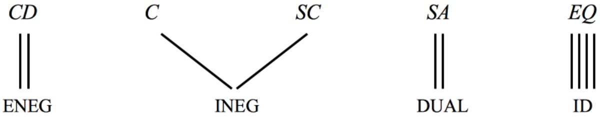
Natural way of extending the square:

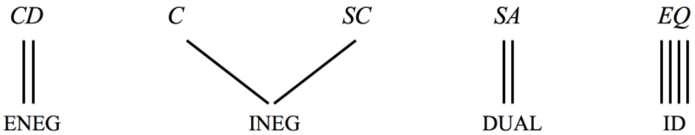
- **adding more independent negation positions**
- i.e. adding more copies of $\mathbb{Z}_2$
- logically: from $\mathbb{Z}_2 \times \mathbb{Z}_2$ to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
  from 2 negation positions to 3 negation positions
  from $2^2 = 4$ duality functions to $2^3 = 8$ duality functions
- geometrically: from square to cube/octagon to ...



Aristotelian/duality multigraph (ADM): visualizes how many times a specific combination of Aristotelian and duality relation occurs in the square

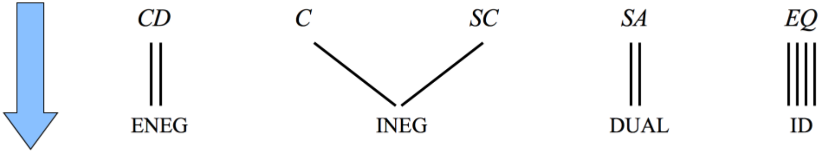






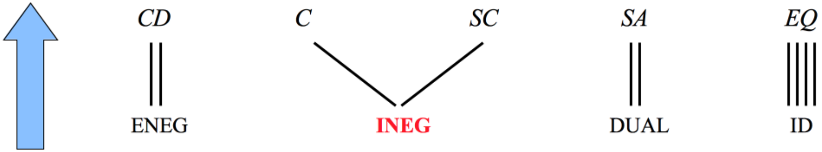
The correspondence between Aristotelian and duality relations is not perfect, but still highly regular

Aristotelian/Duality Multigraph (ADM)



The correspondence between Aristotelian and duality relations is not perfect, but still highly regular

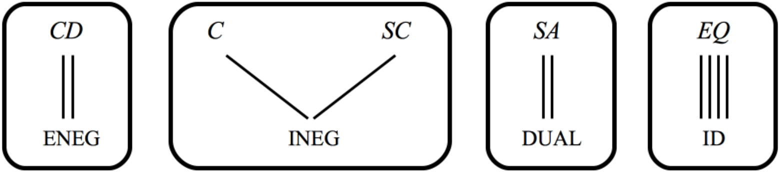
- each Aristotelian relation corresponds to a unique duality relation



The correspondence between Aristotelian and duality relations is not perfect, but still highly regular

- each Aristotelian relation corresponds to a unique duality relation
- vice versa, duality relations
  - ENEG, DUAL and ID correspond to a unique Aristotelian relation
  - INEG corresponds to two Aristotelian relations

Aristotelian/Duality Multigraph (ADM)



The correspondence between Aristotelian and duality relations is not perfect, but still highly regular

- each Aristotelian relation corresponds to a unique duality relation
- vice versa, duality relations:
  - ENEG, DUAL and ID correspond to a unique Aristotelian relation
  - INEG corresponds to two Aristotelian relations
- ADM for the square of opposition has 4 **connected components**, viz. $\{CD, \text{ENEG}\}$, $\{C, SC, \text{INEG}\}$, $\{SA, \text{DUAL}\}$ and $\{EQ, \text{ID}\}$

- this close correspondence leads to a **quasi-identification** of the two types of squares:
  - using Aristotelian terminology to describe duality square (or vice versa)
  - viewing one as a generalization of the other
  - already noted in medieval logic (Peter of Spain, William of Sherwood):
    - mnemonic rhyme: *pre contradic, post contra, pre postque subalter*
    - ENEG = *pre* $\approx$ *CD*, INEG = *post* $\approx$ *C*, DUAL = *pre postque* $\approx$ *SA*

- still some **crucial differences**:
  - duality relations are all **symmetric** $\Leftrightarrow$ Aristotelian *SA* is asymmetric
  - duality relations are all **functional** $\Leftrightarrow$ Aristotelian *C*, *SC* and *SA* are not

    ☞ Löbner (1990, 2011), Peters & Westerståhl (2006), Westerståhl (2012)
  - duality relations are **not logic-sensitive** $\Leftrightarrow$ Aristotelian relations are

    ☞ lecture 4

- the most powerful way to argue for the independence of Aristotelian and duality diagrams consists in analyzing diagrams **beyond** the square
- the **hexagon** is not the most natural extension of the square:
  - natural extension from Aristotelian perspective (6 is a multiple of 2)
  - not natural extension from duality perspective (6 is not a power of 2)

  $\Rightarrow$ JSB and SC hexagon are **less informative** than classical square
- **octagon** = natural extension from Aristotelian + duality perspective:

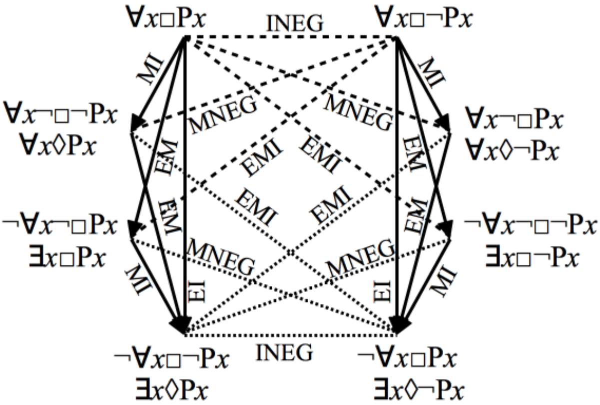
from	**square**	to	**octagon**		
	$2 \times 2 = 4 = 2^2$		$4 \times 2 = 8 = 2^3$		
	2 PCDs $\leftrightsquigarrow 2 \times 2$		4 PCDs $\leftrightsquigarrow 4 \times 2$	$\Rightarrow$	Aristotelian view
	2 negations $\leftrightsquigarrow 2^2$		3 negations $\leftrightsquigarrow 2^3$	$\Rightarrow$	duality view

- discuss four octagons in detail:
  - **four** different **Aristotelian families** of octagons
  - **two** types of **generalised duality**, each revealed in two octagons

- suppose that φ is the result of applying an n-ary composed operator $O_1 \circ O_2$ to n propositions $\alpha_1, \ldots, \alpha_n$
- $\varphi \equiv (O_1 \circ O_2)(\alpha_1, \ldots, \alpha_n) = O_1(O_2(\alpha_1, \ldots, \alpha_n))$
- an extra negation position has become available!

- the proposition $O_1(O_2(\alpha_1, \ldots, \alpha_n))$ has a unique
  - **external negation** (ENEG): $\neg O_1(O_2(\alpha_1, \ldots, \alpha_n))$
  - **intermediate negation** (MNEG): $O_1(\neg O_2(\alpha_1, \ldots, \alpha_n))$
  - **internal negation** (INEG): $O_1(O_2(\neg\alpha_1, \ldots, \neg\alpha_n))$

- 3 independent negation positions $\Rightarrow 2^3 = 8$ duality functions in total
- much richer duality behavior:
  - ENEG, MNEG, and INEG
  - ENEG $\circ$ MNEG (EM), ENEG $\circ$ INEG (EI), and MNEG $\circ$ INEG (MI)
  - ENEG $\circ$ MNEG $\circ$ INEG (EMI)


∀x□Px
INEG
∀x□¬Px
MI
MI
∀x¬□¬Px
∀x◊Px
MNEG
MNEG
∀x¬□Px
∀x◊¬Px
EM
EMI
EMI
EM
¬∀x¬□Px
∃x□Px
EM
EMI
EMI
EM
¬∀x¬□¬Px
∃x□¬Px
MNEG
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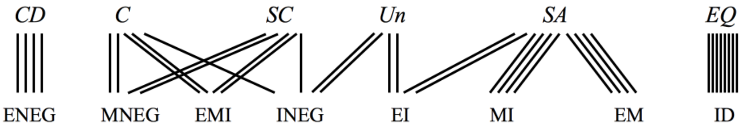
CD
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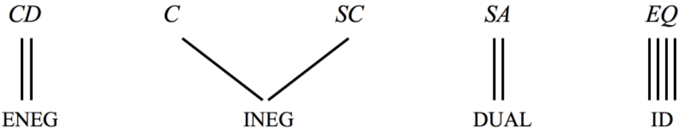
CD
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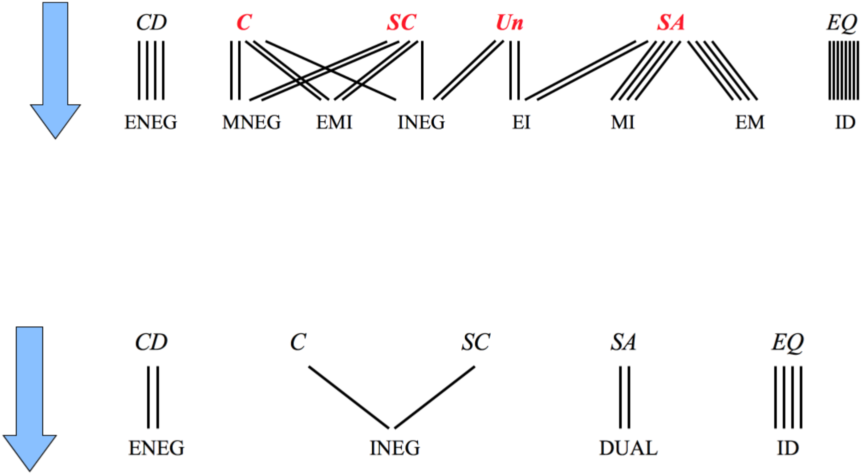
DUAL
ID

Buridan octagon (modal syllogistics)



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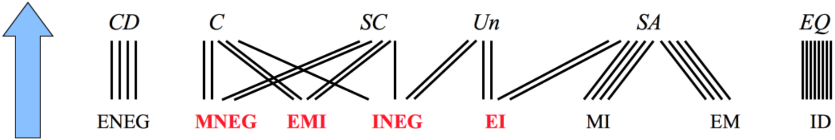
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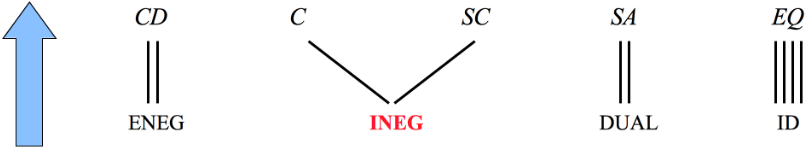
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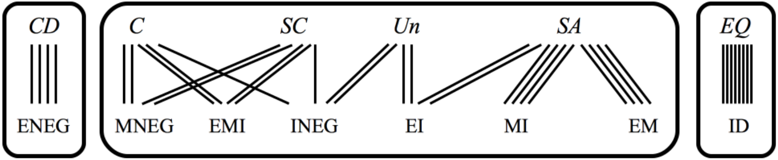
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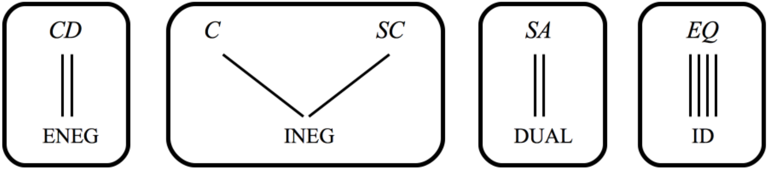
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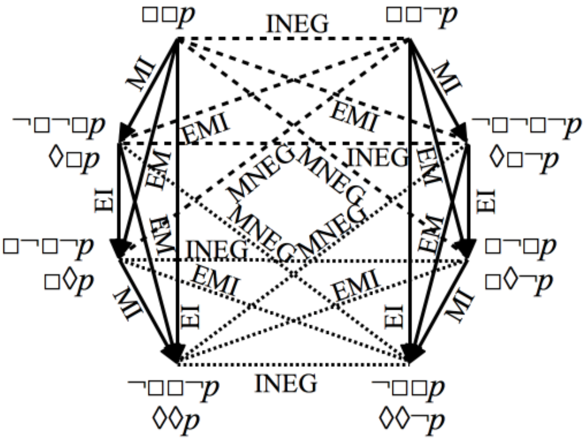
□□p
INEG
□□¬p
MI
MI
¬□¬□p
◊□p
EMI
EMI
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◊□¬p
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MNEG
MNEG
INEG
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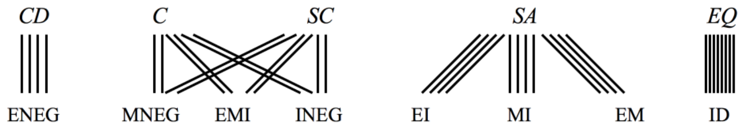

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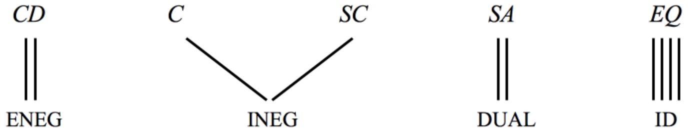

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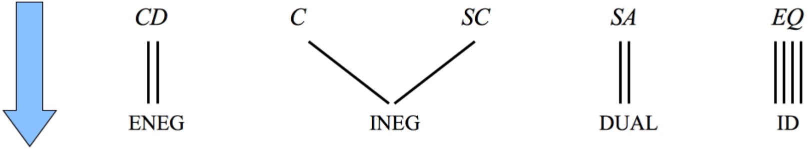
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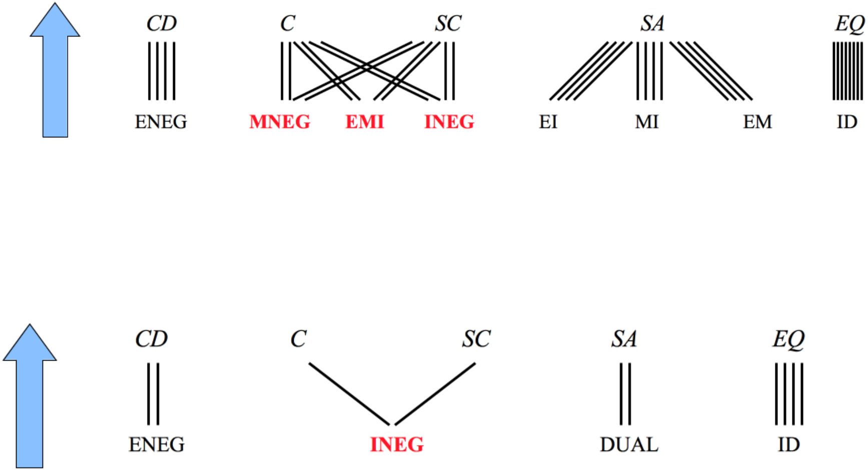
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DUAL
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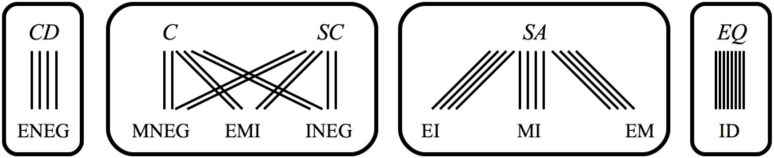
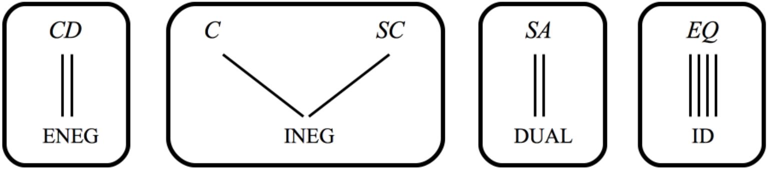
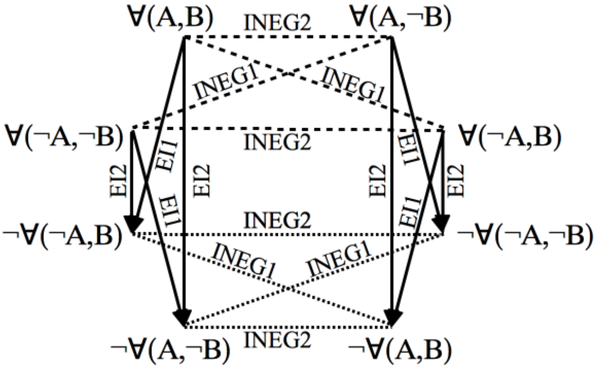
Lenzen octagon (modal logic S4.2)





- classical duality applies internal negation to **all** arguments, i.e. the internal negation of $O(\alpha_1, \ldots, \alpha_n)$ is $O(\neg\alpha_1, \ldots, \neg\alpha_n)$
- now: apply internal negation to each argument **independently**
- with a binary operator O, we thus have 3 independent negation positions in total: the proposition $O(\alpha_1, \alpha_2)$ has a unique:
  - **external negation** ($\textsc{eneg}$): $\neg O(\alpha_1, \alpha_2)$
  - **first internal negation** ($\textsc{ineg}1$): $O(\neg\alpha_1, \alpha_2)$,
  - **second internal negation** ($\textsc{ineg}2$): $O(\alpha_1, \neg\alpha_2)$
- 3 independent negation positions $\Rightarrow 2^3 = 8$ duality functions in total
- much richer duality behavior:
  - $\textsc{eneg}$, $\textsc{ineg}1$, and $\textsc{ineg}2$
  - $\textsc{eneg} \circ \textsc{ineg}1$ ($\textsc{ei}1$), $\textsc{eneg} \circ \textsc{ineg}2$ ($\textsc{ei}2$), and $\textsc{ineg}1 \circ \textsc{ineg}2$ ($\textsc{i}12$)
  - $\textsc{eneg} \circ \textsc{ineg}1 \circ \textsc{ineg}2$ ($\textsc{ei}12$)


∀(A,B)
INEG2
∀(A,¬B)
INEG1
INEG1
∀(¬A,¬B)
INEG2
∀(¬A,B)
EI1
EI2
EI2
EI1
EI2
EI2
EI1
EI1
¬∀(¬A,B)
INEG2
¬∀(¬A,¬B)
INEG1
INEG1
¬∀(A,¬B)
INEG2
¬∀(A,B)

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CD
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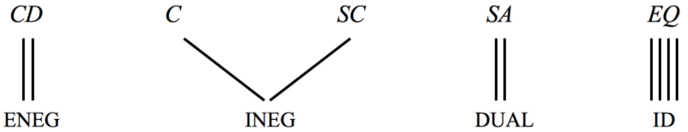
SC
SA
Un
EQ
ENEG
INEG1
INEG2
EI1
EI2
I12
EI12
ID


CD
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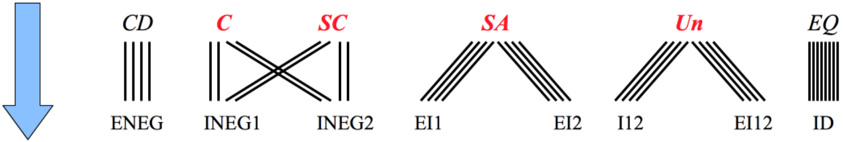
KU LEUVEN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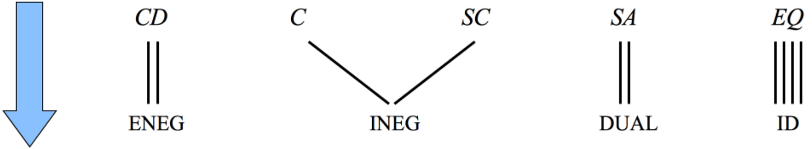
CD
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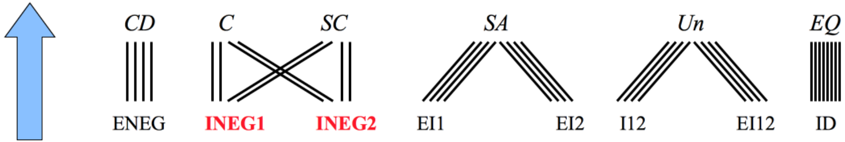
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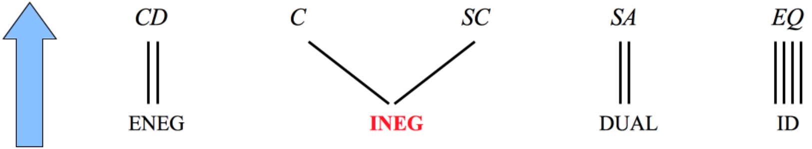
CD
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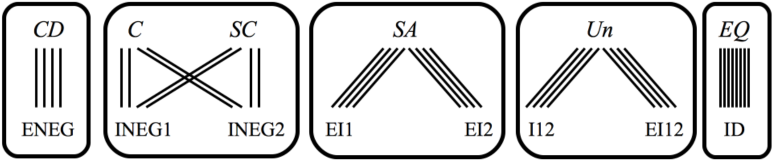
CD
ENEG
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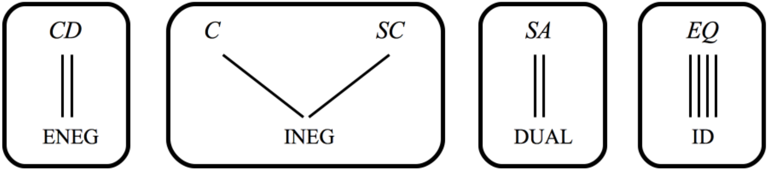
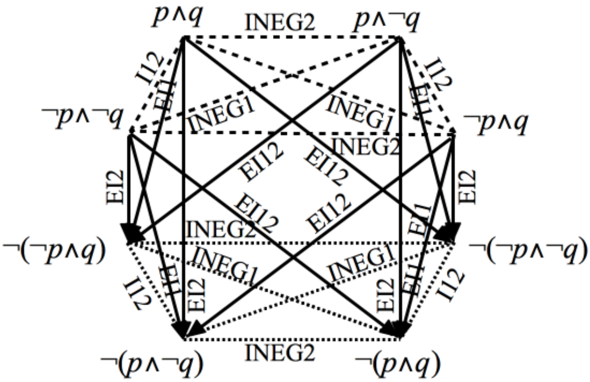
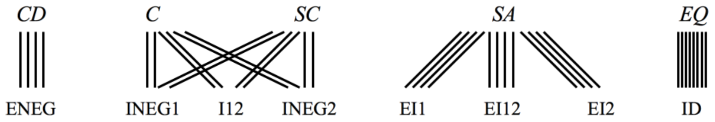

CD
ENEG
C
SC
INEG
SA
DUAL
EQ
ID


$p \wedge q$
INEG2
$p \wedge \neg q$
II2
EI1
INEG1
INEG1
II2
EI1
$\neg p \wedge \neg q$
$\neg p \wedge q$
INEG2
EI12
EI12
EI2
EI2
EI12
EI12
EI1
INEG2
$\neg(\neg p \wedge q)$
$\neg(\neg p \wedge \neg q)$
INEG1
INEG1
II2
EI1
EI2
EI2
EI1
II2
INEG2
$\neg(p \wedge \neg q)$
$\neg(p \wedge q)$


CD
C
SC
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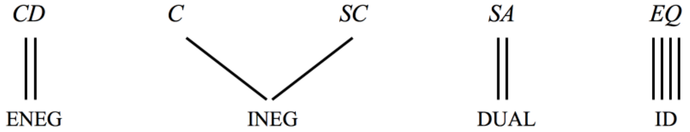
EQ
ENEG
INEG1
I12
INEG2
EI1
EI12
EI2
ID


CD
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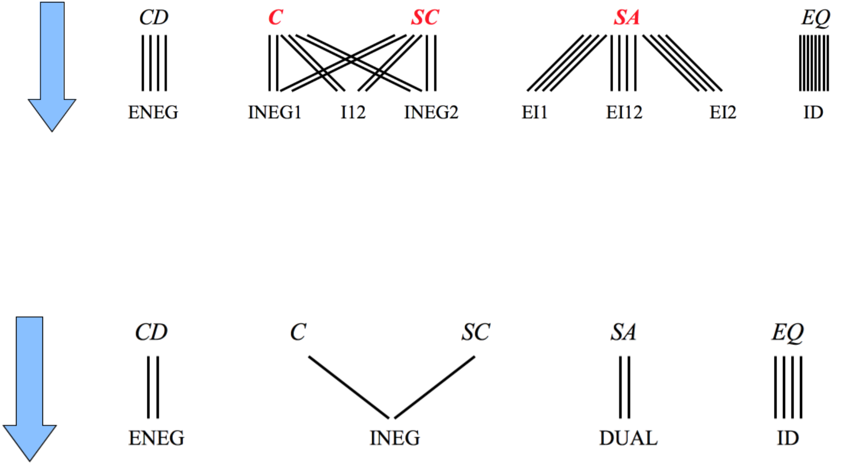

CD
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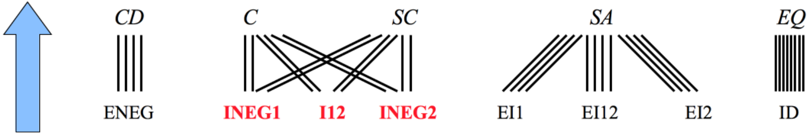
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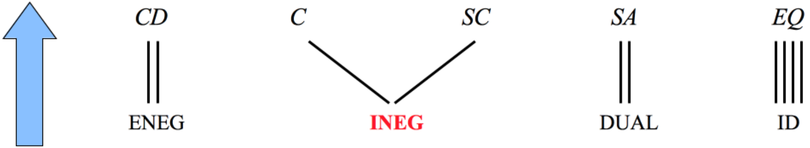
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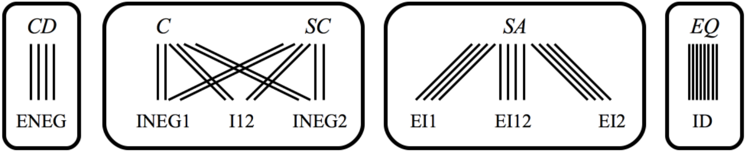
CD
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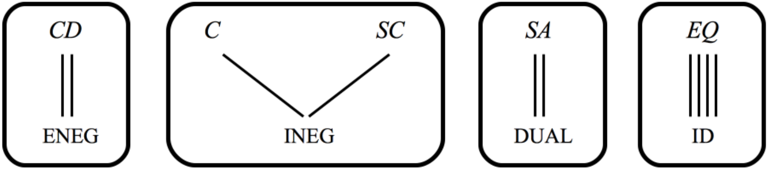
CD
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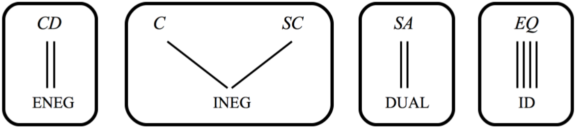
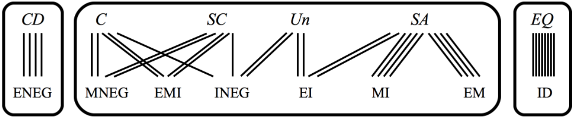
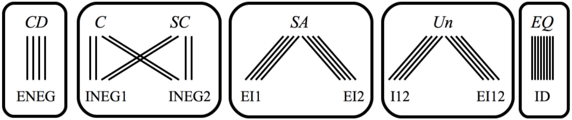
square of opposition ⇝ classical duality



Buridan octagon ⇝ composed operator duality

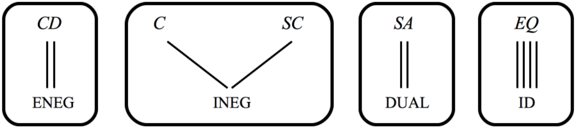


Keynes-Johnson octagon ⇝ generalised Post duality

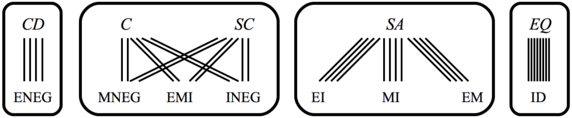


Conclusion

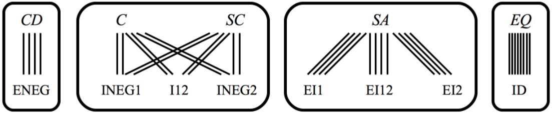
square of opposition ⇝ classical duality



Lenzen octagon ⇝ composed operator duality



Moretti octagon ⇝ generalised Post duality



Thank you! Questions?

More info: `www.logicalgeometry.org`