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Aristotelian and Duality Relations with Proportional Quantifiers

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Fifth World Congress on the Square of Oppositions

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- The central aim of the presentation is to chart which logical relations hold between quantificational formulas expressing the notion of *proportionality*.

- Two families of logical relations:
  - **Aristotelian** relations of contradiction, (sub)contrariety and subalternation
  - **Duality** relations of external, internal and dual negation

- Two types of expressions:
  - **explicit proportionals**: the proportion is explicitly referred to in terms of fractions or percentages:
    - *At least two thirds of the students passed the test.*
    - *Less than 20 percent of the students passed the test.*
  - **implicit proportionals**: the actual proportion remains implicit:
    - *A/the minority/majority of the students passed the test.*

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1. Introduction
2. Aristotelian and Duality Relations
3. Classical versus degenerate Aristotelian and Duality Squares
4. Aristotelian and Duality Squares for Proportional Quantifiers
5. Conclusion

This talk is based on joint work with Lorenz Demey.

Two propositions are:

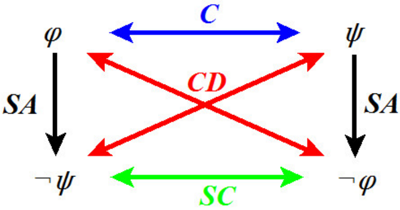
***contradictory* (CD)**	iff	they cannot be true together and they cannot be false together,
***contrary* (C)**	iff	they cannot be true together but they can be false together,
***subcontrary* (SC)**	iff	they can be true together but they cannot be false together,
***in subalternation* (SA)**	iff	the first proposition entails the second but the second doesn't entail the first

The set of Aristotelian relations is fundamentally *hybrid*:

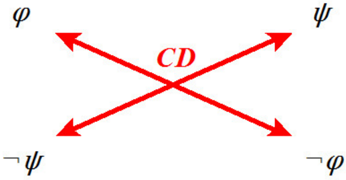
- CD, C and SC are symmetric; definition $\sim$ being true/false together
  SA is not symmetric; definition $\sim$ truth propagation.
- CD is a functional relation, but C, SC and SA are not.
- Smessaert & Demey (2014)

Any fragment of 4 formulas from a logical language $\mathcal{L}$ for a logical system S which is closed under negation (i.e. which consists of two pairs of contradictories) yields an *Aristotelian square* which is

$$\begin{aligned} \textbf{classical} &\equiv (2 \times \mathrm{CD}) + (2 \times \mathrm{SA}) + (1 \times \mathrm{C}) + (1 \times \mathrm{SC}) \\ \textbf{degenerate} &\equiv (2 \times \mathrm{CD}) \end{aligned}$$



classical Aristotelian square



degenerate Aristotelian square

The n-ary connectives/operators O_1 and O_2 are one another's:

***external negation* (EN)**	iff	for all $\varphi_1, \ldots, \varphi_n$ $O_2(\varphi_1, \ldots, \varphi_n) \equiv \neg O_1(\varphi_1, \ldots, \varphi_n)$
***internal negation* (IN)**	iff	for all $\varphi_1, \ldots, \varphi_n$ $O_2(\varphi_1, \ldots, \varphi_n) \equiv O_1(\neg\varphi_1, \ldots, \neg\varphi_n)$
***dual negation* (DN)**	iff	for all $\varphi_1, \ldots, \varphi_n$ $O_2(\varphi_1, \ldots, \varphi_n) \equiv \neg O_1(\neg\varphi_1, \ldots, \neg\varphi_n)$

Transpose definitions of EN/IN/DN from *operators* to *formulas*: if operators O_1 and O_2 are each other's EN/IN/DN, then formulas $O_1(\varphi_1 \ldots \varphi_n)$ and $O_2(\varphi_1 \ldots \varphi_n)$ are said to be each other's EN/IN/DN as well.

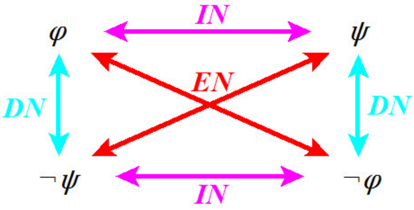
The set of duality relations is fundamentally *uniform*:

- EN, IN and DN are all symmetric relations.
- EN, IN and DN are all functional relations.

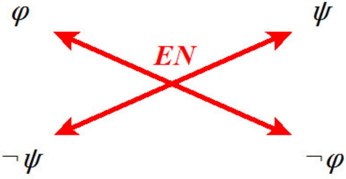
Duality squares

Any fragment of 4 formulas from a logical language $\mathcal{L}$ for a logical system S which is closed under negation (i.e. which consists of two pairs of contradictories) yields a *duality square* which is

$$
\begin{array}{rcl}
\textbf{classical} & \equiv & (2 \times \mathrm{EN}) + (2 \times \mathrm{IN}) + (2 \times \mathrm{DN}) \\
\textbf{degenerate} & \equiv & (2 \times \mathrm{EN})
\end{array}
$$

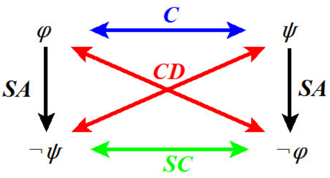


classical duality square

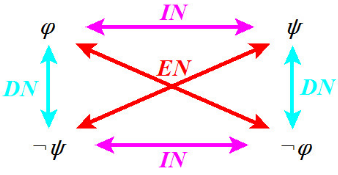


degenerate duality square

- Löbner (1990,2011), Peters & Westerståhl (2006), Westerståhl (2012), Demey (2012), Smessaert (2012).
- All duality relations are symmetric but not all Aristotelian relations are.
- All duality relations are functional but not all Aristotelian relations are.
- The duality relation IN corresponds to Aristotelian C and/or SC.
- Aristotelian relations are highly logic-sensitive, whereas duality relations are insensitive to underlying logic: Demey (2015), Demey & Smessaert (2016).



classical Aristotelian square



classical duality square

The functions ID, ENEG, INEG and DUAL jointly form a group that is isomorphic to the *Klein four group* $\mathbf{V}_4$. Its Cayley table looks as follows:

$\circ$	ID	ENEG	INEG	DUAL
ID	ID	ENEG	INEG	DUAL
ENEG	ENEG	ID	DUAL	INEG
INEG	INEG	DUAL	ID	ENEG
DUAL	DUAL	INEG	ENEG	ID

The functions ID, ENEG, INEG and DUAL jointly form a group that is isomorphic to the *Klein four group* $\mathbf{V}_4$. Its Cayley table looks as follows:

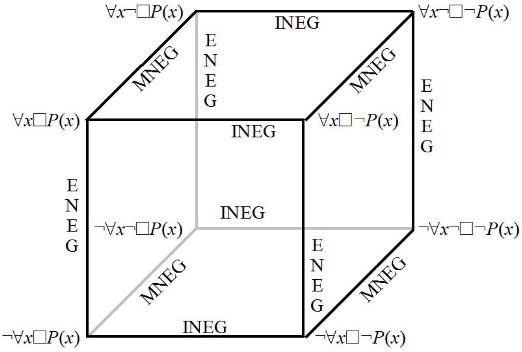
$\circ$	ID	ENEG	INEG	DUAL
ID	ID	ENEG	INEG	DUAL
ENEG	ENEG	ID	DUAL	INEG
INEG	INEG	DUAL	ID	ENEG
DUAL	DUAL	INEG	ENEG	ID

$\mathbf{V}_4$ is isomorphic to the direct product of $\mathbb{Z}_2$ with itself, i.e. $\mathbf{V}_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. The Cayley table for $\mathbb{Z}_2 \times \mathbb{Z}_2$ looks as follows:

$\circ$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$(0,0)$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$(1,0)$	$(1,0)$	$(0,0)$	$(1,1)$	$(0,1)$
$(0,1)$	$(0,1)$	$(1,1)$	$(0,0)$	$(1,0)$
$(1,1)$	$(1,1)$	$(0,1)$	$(1,0)$	$(0,0)$

generalisation to multiple/combined operators

- from 2 negation positions to 3 negation positions
- **ENEG** OPERATOR1 **MNEG** OPERATOR2 **INEG**
- from $\mathbb{Z}_2 \times \mathbb{Z}_2$ to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$



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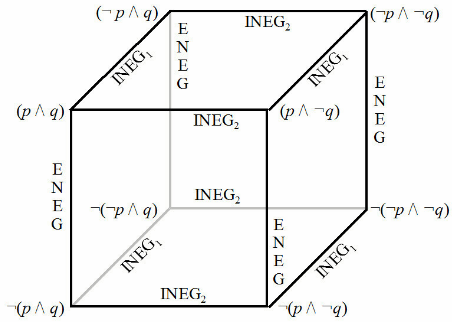
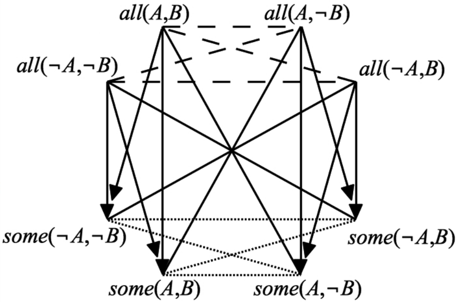
Generalized Post-duality

- from 2 negation positions to 3 negation positions
- **ENEG** OPERATOR **INEG1 INEG2**
- from $\mathbb{Z}_2 \times \mathbb{Z}_2$ to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$



propositional connectives



Keynes-Johnson octagon

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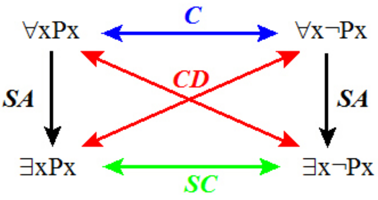
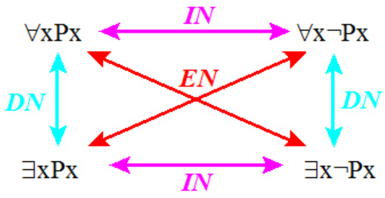
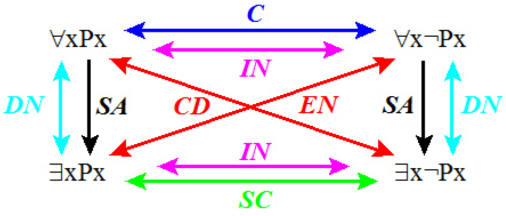


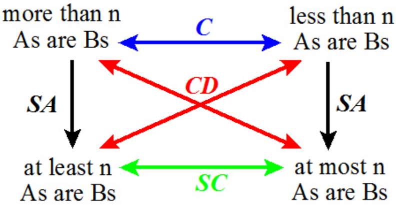
classical Aristotelian square



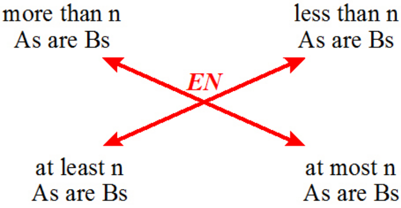
classical duality square



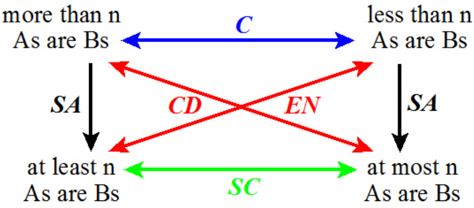
quantifiers of standard First Order Logic



classical Aristotelian square

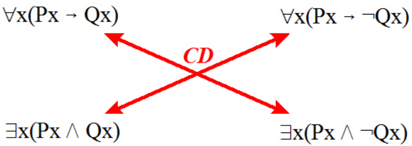


degenerate duality square

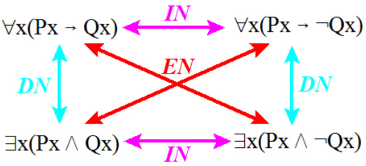


The numerical quantifiers

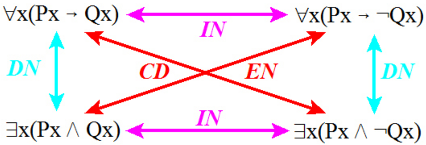
Type 3 square: degenerate Aristotelian + classical Dual



degenerate Aristotelian square

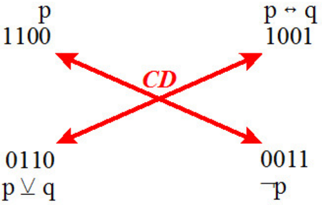


classical duality square

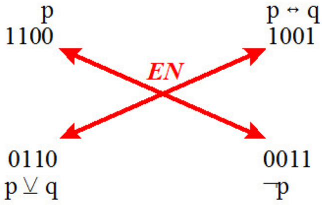


categorical statements in
standard First Order Logic (no EI)

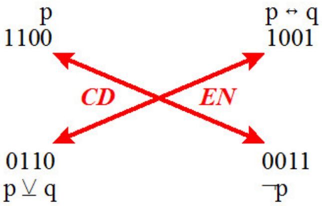
Type 4 square: degenerate Aristotelian + degenerate Dual



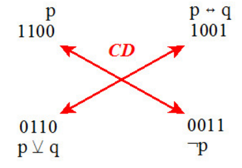
degenerate Aristotelian square



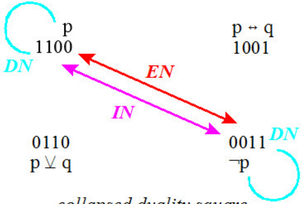
degenerate duality square



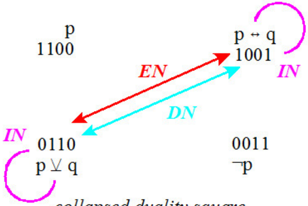
Propositional connectives



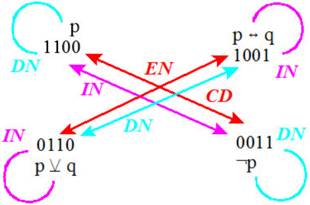
degenerate Aristotelian square



collapsed duality square
(self dual negation)



collapsed duality square
(self internal negation)



Propositional connectives

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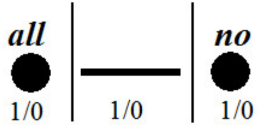
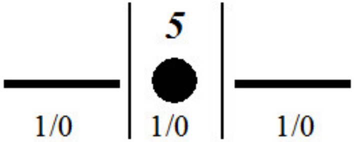
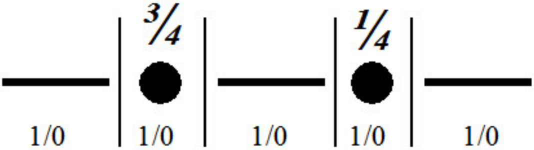
Scales and Bitstrings for Numerical and Standard Quantifiers



100	all	011	not all
010	some but not all	101	no or all
001	no	110	some

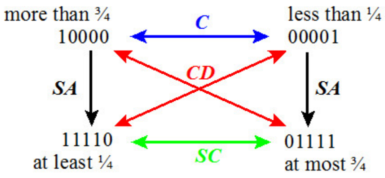


100	more than 5	011	at most 5
010	exactly 5	101	not exactly 5
001	less than 5	110	at least 5

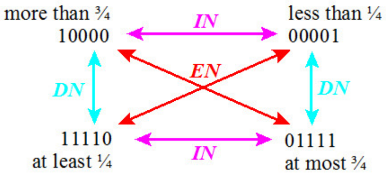


10000	more than 3/4	01111	at most 3/4
01000	exactly 3/4	10111	not exactly 3/4
00100	less t. 3/4 **but** more t. 1/4	11011	at least 3/4 **or** at most 1/4
00010	exactly 1/4	11101	not exactly 1/4
00001	less than 1/4	11110	at least 1/4

11000	more than 3/4 **or** exactly 3/4	$\equiv$	at least 3/4
00011	less than 1/4 **or** exactly 1/4	$\equiv$	at most 1/4
01110	at most 3/4 **but** at least 1/4	$\equiv$	between 1/4 and 3/4

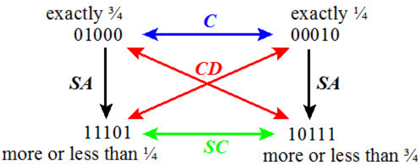


classical Aristotelian square

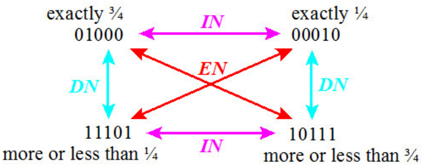


classical duality square

10000		*More than 3/4 of the students passed the test.*
	$\equiv$	*Less than 1/4 of the students failed the test.*
00001		*Less than 1/4 of the students passed the test.*
	$\equiv$	*More than 3/4 of the students failed the test.*
11110		*At least 1/4 of the students passed the test.*
	$\equiv$	*At most 3/4 of the students failed the test.*
01111		*At most 3/4 of the students passed the test.*
	$\equiv$	*At least 1/4 of the students failed the test.*



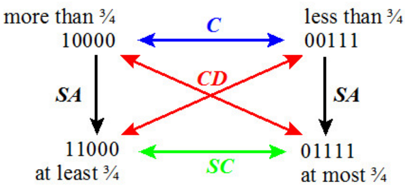
classical Aristotelian square



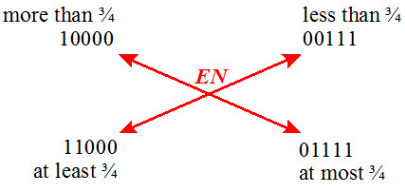
classical duality square

01000		*Exactly 3/4 of the students passed the test.*
	$\equiv$	*Exactly 1/4 of the students failed the test.*
00010		*Exactly 1/4 of the students passed the test.*
	$\equiv$	*Exactly 3/4 of the students failed the test.*
11101		*More or less than 1/4 of the students passed the test.*
	$\equiv$	*More or less than 3/4 of the students failed the test.*
10111		*More or less than 3/4 of the students passed the test.*
	$\equiv$	*More or less than 1/4 of the students failed the test.*

Type 2 square: classical Aristotelian + degenerate Dual



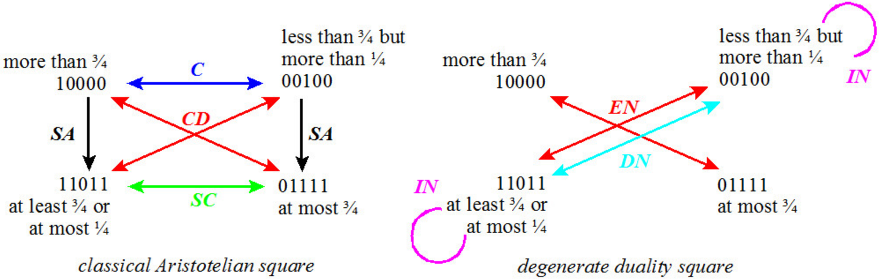
classical Aristotelian square



degenerate duality square

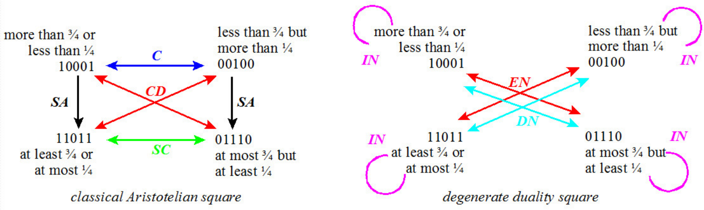
10000	*More than 3/4 of the students passed the test.*
00111	*Less than 3/4 of the students passed the test.*
11000	*At least 3/4 of the students passed the test.*
01111	*At most 3/4 of the students passed the test.*

single collapse with self-internal negation



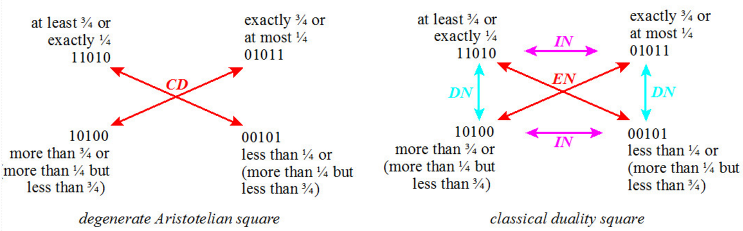
10000		*More than 3/4 of the students passed the test.*
00100		*Less than 3/4 but more than 1/4 of t.s. passed the test.*
	≡	*Less than 3/4 but more than 1/4 of t.s. failed the test.*
11011		*At least 3/4 or at most 1/4 of the students passed the test.*
	≡	*At least 3/4 or at most 1/4 of the students failed the test.*
01111		*At most 3/4 of the students passed the test.*

double collapse with self-internal negation



10001		*More than 3/4 or less than 1/4 of t.s. passed the test.*
	$\equiv$	*More than 3/4 or less than 1/4 of t.s. failed the test.*
00100		*Less than 3/4 but more than 1/4 of t.s. passed the test.*
	$\equiv$	*Less than 3/4 but more than 1/4 of t.s. failed the test.*
11011		*At least 3/4 or at most 1/4 passed.*
	$\equiv$	*At least 3/4 or at most 1/4 failed.*
01110		*At most 3/4 but at least 1/4 of t.s. passed.*
	$\equiv$	*At most 3/4 but at least 1/4 of t.s. failed.*

Type 3 square: degenerate Aristotelian + classical Dual



11010		*At least 3/4 or exactly 1/4 of t.s. passed the test.*
	≡	*At most 1/4 or exactly 3/4 of t.s. failed the test.*
01011		*At most 1/4 or exactly 3/4 of t.s. passed the test.*
	≡	*At least 3/4 or exactly 1/4 of t.s. failed the test.*
10100		*More than 3/4 or more than 1/4 but less than 3/4 passed.*
	≡	*Less than 1/4 or more than 1/4 but less than 3/4 failed.*
00101		*Less than 1/4 or more than 1/4 but less than 3/4 passed.*
	≡	*More than 3/4 or more than 1/4 but less than 3/4 failed.*

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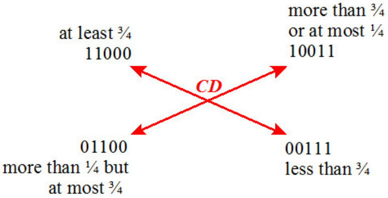
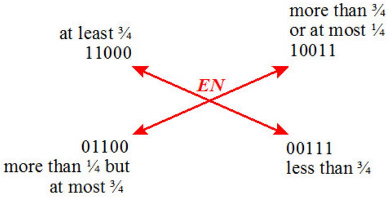


degenerate Aristotelian square



degenerate duality square

11000	*At least 3/4 of the students passed the test.*
10011	*More than 3/4 or at most 1/4 of the students passed the test.*
01100	*At most 3/4 but more than 1/4 of the students passed the test.*
00111	*Less than 3/4 of the students passed the test.*

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- *A/the majority of the students passed the test.*
- *A/the minority of the students passed the test.*
- *Less than a/the majority of the students passed the test.*
- *More than a/the minority of the students passed the test.*
- *At least a/the majority passed the test.* => ?probably all students
- *At most a/the minority passed the test.* => ?probably no students

- **Exactly a/the majority of the students passed the test.*
- **Exactly a/the minority of the students passed the test.*
- *??More than a/the majority passed the test.* => ??all students
- *??Less than a/the minority passed the test.* => ??no students
- *?At most a/the majority passed the test.* => ??not all students
- *?At least a/the minority passed the test.* => ??some students

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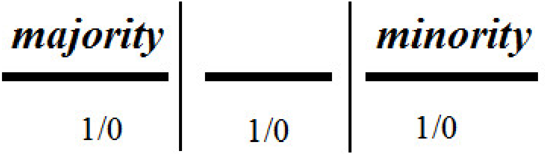
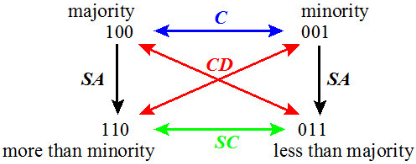
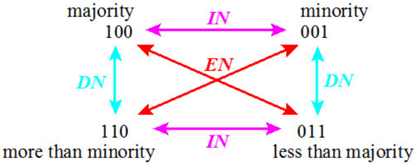


100	a majority	011	not a majority / less than a majority
010	not a majority **but** not a minority	101	a majority **or** a minority
001	a minority	110	not a minority / more than a minority

Type 1 square: classical Aristotelian + classical Dual



classical Aristotelian square



classical duality square

100		*A majority of the students passed the test.*
	$\equiv$	*A minority of the students failed the test.*
001		*A minority of the students passed the test.*
	$\equiv$	*A majority of the students failed the test.*
110		*More than a minority of the students passed the test.*
	$\equiv$	*Less than a majority of the students failed the test.*
011		*Less than a majority of the students passed the test.*
	$\equiv$	*More than a minority of the students failed the test.*

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Conclusion

- Chart the logical relations between quantificational formulas expressing the notion of *proportionality*.
- Two families of logical squares:
  - **Aristotelian** squares: **two** subtypes: classical vs degenerate
  - **Duality** squares: **more** subtypes
    - two basic subtypes: classical vs degenerate
    - collapsed duality squares with self-internal and self-dual negation
    - singly collapsed versus doubly collapsed duality squares
- Two types of expressions:
  - **explicit proportionals**:
    - *More than/exactly/less than 3/4 of the students passed the test.*
    - bitstrings of **length five**
  - **implicit proportionals**:
    - *A/the minority/majority of the students passed the test.*
    - bitstrings of **length three**

Thank you!

More info: www.logicalgeometry.org

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